Optimal Static Hedging of Volumetric Risk in a Competitive Wholesale Electricity Market *

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Abstract

In competitive wholesale electricity markets, regulated load serving entities (LSEs) and marketers with default service contracts have obligations to serve fluctuating load at predetermined fixed prices while meeting their obligation through combinations of long-term contracts, wholesale purchases and self-generation that are subject to volatile prices or opportunity cost. Hence, their net profits are exposed to joint price and quantity risk both of which are correlated with weather variations. In this paper we develop a static hedging strategy for the LSE (or marketer) whose objective is to minimize a mean-variance utility function over net profit, subject to a self-financing constraint. Since quantity risk is non-traded, the hedge consists of a portfolio of price-based financial energy instruments, including a bond, forward contract and a spectrum of European call and put options with various strike prices. The optimal hedging strategy is jointly optimized with respect to contracting time and the portfolio mix, which varies with contract timing, under specific price and quantity dynamics and the assumption that the hedging portfolio which matures at the time of physical energy delivery is purchased at a single point in time. Explicit analytical results are derived for the special case where price and

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quantity have a joint bivariate lognormal distribution.

Key Words: Energy Risk, Competitive Electricity Markets, Volumetric Hedging, Incomplete Markets

1 Introduction

Electricity is one of the most (if not the most) volatile traded commodity.[1] During the summer of 1998, wholesale power prices in the Midwest of the U.S. surged to a stunning amount of $7,000 per MWh from the normal price range of $30 ∼ $60 per MWh causing the defaults of two power marketers on the east coast. In February 2004, persistent high prices in Texas during an ice storm that lasted three days rose to $1000/MWh and led to the bankruptcy of a retail energy provider that was exposed to spot market prices. More recently on January 16, 2007 high temperatures across New South Wales, Victoria and South Australia saw the demand for electricity reach record highs, resulting in electricity spot prices in excess of $5000/MWh. In California during the 2000/2001 electricity crisis wholesale spot prices rose sharply and persisted around $500 per MWh. The devastating economic consequences of that crisis were largely attributed to the fact that the major utilities, who were forced to sell power to their customers at low fixed prices set by the regulator, were not properly hedged through long-term supply contracts. Such expensive lessons have raised the awareness of market participants to the importance and necessity of risk management in competitive electricity markets.

To load-serving entities (LSEs) in electricity markets, such evident price risk is not the only market risk they face. Since electricity customers are free to control their consumption with a flip of a switch, LSEs are also uncertain about how much electricity their customers will use at a certain hour until the customers actually turn switches on and draw electricity. Uncertainty or unpredictability of demand is a traditional concern for any commodity, but holding inventory is a good solution to mitigate quantity risk for those commodities that can be economically stored. However, electricity is non-storable, which is the most important characteristic that differentiates the

[1]Typical volatilities: dollar/yen exchange rates (10%-20%), LIBOR rates (10%-20%), SP500 index (20%-30%), NASDAQ (30%-50%), natural gas prices (50%-100%), and spot electricity (100%-500% and higher) [EW03].

electricity market from the money market or other commodities markets. Since electricity needs to be generated at the same time it is consumed, the traditional method of purchasing an excess quantity of a product when prices are low and holding inventories cannot be used by firms retailing electricity. Moreover, unlike other commodities, LSEs, which are typically regulated, operate under an obligation to serve and cannot curtail service to their customers (except under special service agreement) or pass through high wholesale prices to their customers by charging more when they cannot procure electricity at favorable prices.[2] Consequently, LSEs may have to procure spot electricity to meet the demand at prevailing spot prices, which which could exceed the retail prices paid by their customers.

A similar problem is faced by marketers or generating companies that sold load-following fixed price contract in the default service auction in New Jersey. LSEs in New Jersey were ordered by the local public utility commissions to procure default service contracts from generators via auction.[LS04] The sellers of such contracts assume an obligation to provide a proportional slice of the fluctuating load at a fixed energy price set through an auction. Generators or marketers selling such contracts have their profit (if they procure in the wholesale market) or their opportunity cost (if they sel-generate) exposed to price and volumetric risks.

Fluctuations in electricity demand also affects the spot price itself. Demand uncertainty can cause volatile and spiky spot prices and thus may result in high uncertainty in LSEs' profits, because profits (costs and earnings, too) are a function of electricity demand multiplied by the spot price. Furthermore, the volumetric risks in the electricity market become severe due to adverse movements of wholesale price and demand; for instance, the sales volume is small when the profit margin is high, while it is large when the margin is low or even negative. This is due to the price inelasticity of demand and the resulting strong positive correlation between spot price and demand. Thus, volumetric risk management should always include the price risk, and vice versa, and hedging strategies that focus only on price risks for a fixed amount of volume cannot fully hedge market risks faced by LSEs.

In this paper, we develop a static hedging strategy that deals with price and volumetric risk. Specifically, we develop a hedging portfolio of standard financial instruments for electric power, such as forward, European calls and

[2]In fact, most of US states which opened their retail markets to competition have frozen their retail electricity prices.

puts and co-optimize the portfolio mix and procurement time of the contracts.

While weather derivatives, whose underlying is strongly correlated with electricity demand, can be also effective alternatives in hedging volumetric risks, we do not include such instruments in our hedging portfolio. The speculative image of the weather derivatives makes them undesirable for a regulated utility having to justify to a regulator its risk management practices and the cost associated with such practices (which are passed on to customers). Moreover, weather derivatives cannot insure supply adequacy, which is a major concern in the electricity industry. In some jurisdictions, the regulators (e.g. the California Public Utility Commission (CPUC)) who are motivated by concerns for generation adequacy, require that LSEs hedge their load serving obligations and appropriate reserves with physically covered electricity forward contracts and options. That is, the hedges cannot be settled financially or subject to liquidation damages but must be covered by specific installed or planned generation capacity capable of physical delivery. In California, the CPUC has explicitly ordered phasing out of financial contracts with liquidation damages as means of meeting generation adequacy requirement by 2008. [CPU04a, CPU04b] On the other hand private marketer with load following obligations may find wether derivatives attractive.

Our earlier paper [OOD06] devoted to constructing the optimal portfolio which consists of electricity derivatives such as forwards and calls and puts of different strikes. Specifically, we obtained the optimal hedging strategy that uses electricity derivatives to hedge price and volumetric risks by maximizing the expected utility of the hedged profit. When such a portfolio is held by an LSE, the call options with strikes being below the spot price will be exercised so that the amount of the options being exercised is procured at the strike prices. Using this strategy, the LSE can set an increasing price limit on incremental load by paying the premiums for the options. This strategy is not only effective in managing quantity risk but was also suggested in the market design literature such as Chao and Wilson [CW04], Oren [Ore05], and Willems [Wil06] as means to achieve resource adequacy, mitigate market power, and reduce spot price volatility.

The analysis in [OOD06] assumed a single time period setting where the hedging portfolio is procured at the beginning of the period and exercised at the end of the period when electricity is delivered. This paper extends our previous work by allowing contract procurement to take place anywhere between the decision time at the onset of the period and the exercise time at the end of the period (when delivery occurs)as long as the entire hedging portfolio is procured at a single point in time. Within this framework we co-optimize the mix and procurement time of the hedging portfolio.

In Section 3 we first solve for the optimal payoff of a general static hedging function given the procurement time, and then finds a replicating portfolio that consists of forward, European calls and puts which yields the optimal payoff. Forward and options prices that are included in our hedging portfolio change as the time approaches delivery time, reflecting the changing expectations in the market. Thus, the mix of the optimal hedging portfolio also changes with the hedging time. Our result shows that hedging too late can increase risk sharply. Optimizing such timing decisions requires solving an integrated problem of selecting the optimal hedging portfolio and time. This problem is considered in Section 4.

This work contributes to scarce literature on volumetric hedging and the hedging-timing decision. At this point in time the practical value of our results is somewhat limited due to the limited availability of electricity derivatives with a full spectrum of strike prices. As more and more marketers or traders are coming into the electricity market, making electricity instruments more liquid the hedging strategies developed in this paper will become more relevant in practice.

We start with Section 2 where we present a summary of relevant studies and move to Section 3 where we solve for the optimal hedging portfolio and Section 4 where we solve for the optimal hedging time. The paper concludes with Section 5.

2 Literature review

In this section we first review the literature addressing the problem of the problem of hedging non-traded risk in an incomplete market. Then we will discuss the limited literature that deals directly with the problem of hedging price and quantity risk in the context of electric power markets.

2.1 Hedging Nontraded Quantity Risk

Conventional hedging strategies typically deal with a single source of uncertainty which is traded in the market through a variety of financial instruments whose payoff are directly linked to the uncertain underlying quantity (e.g., commodity price). In many cases, however, more than one source of uncertainty interacts with each other. One example is risk in the domestic currency value of a foreign stock, where both the foreign exchange rate and the foreign stock price interact with each other. Another example is the hedging problem of a farmer who is uncertain about the output quantity and the selling price at harvest. Such problems can be classified as hedging problems with quantity uncertainty, which can be either traded (e.g. exchange rate) or nontraded (e.g. farmer's output, or LSE's demand).[3] This work focuses on the later case,i.e, the hedging problems with nontraded quantity risk.

The hedging problem with nontraded quantity risk was first dealt with theoretically by McKinnon [McK67]. He recognized that risk aversion for a farmer consists of protecting himself from the output uncertainty as well as the market price uncertainty, and obtained the optimal position of short futures that minimizes the variance of profit. Assuming profit is given by $QS + H(F - S)$ where S is a spot price, Q is an uncertain output, H is the futures position, and F is a futures price ($F = E[S]$), the optimal hedge position is given by

$$H^* = \frac{cov(SQ, S)}{var(S)}.$$

Under the assumption of bivariate normality on spot price and quantity, McKinnon developed an explicit formula for H^* in terms of correlation coefficients and variances of price and quantity. The formula for the optimal hedging quantity showed that the correlation between (production) quantity uncertainty and price uncertainty is a fundamental feature of the problem. Other studies by Danthine [Dan78], Holthausen [Hol79], and Feder et al. [FJS80] extended the McKinnon's work to show that the nontraded quantity risk in the farmer's problem can be partially managed through the optimal choice of the input amount (e.g. the amount of input affects the output quantity) and futures position in a single optimization problem.

The limitation of the models based on McKinnon [McK67] is that they only consider futures contracts as their hedging instruments. When a firm faces a multiplicative risk of price and quantity, its profit is nonlinear in price. In other words, the risk cannot be fully hedged by a forward or futures contract, which has a linear payoff structure. Moreover, as pointed out by Wong

[3] There is another stream of research on the optimal portfolio choice considering additive, but correlated, nontraded risk (Duffie and Zariphopoulou [DZ93] and He and Pages [HP93]). Instead, the risk we consider in this work is a multiplicative function of traded and nontraded risks.

[Won03], even without quantity risks, forward or futures contracts may not be enough because the nonlinearity may stem from using nonlinear marginal utility functions. Specifically, if the firm's preference satisfies reasonable behavioral assumption of prudence [Kim90, Kim93], the prudent firm will have a convex marginal utility function. Such a firm is more sensitive to low realizations of profit than high ones. To avoid the low realizations, the firm finds the asymmetric payoff profiles of options particularly useful. This is the case even with two independent sources of risk.

This idea was employed by Moschini and Lapan [ML95] who included options, that have nonlinear payoff, in the optimal decisions for firms facing price, quantity, and basis risks under CARA utility. They solved for the optimal amount of straddle[4] as well as futures in their hedging portfolio. They demonstrated that the nontraded quantity uncertainty can provide a rationale for the use of options.

Brown and Toft [BT02] also considered a model accounting for multiplicative interaction of traded price risk and nontraded quantity risk and explored the best exotic option for the customized hedging needs. Brown and Toft derived the optimal payoff function for firm's value based on the payoff (as a function of price) from an exotic derivative. They demonstrated that this optimal exotic derivative better-hedges the firm's price and quantity risks than the simple hedge, which uses a single plain vanilla option.

2.2 Hedging for Load-Serving Entities

Restructuring of electricity markets in recent years introduced inherent high volatility in electricity spot price and resulted in the need for efficient hedging strategies for both generators and LSEs. However, the literature on this subject is scarce.

Vehvilainen and Keppo [VK03] developed an integrated framework for the optimal management of price risk using a portfolio of electricity derivatives. Specifically, they provided a framework for the Monte Carlo simulation procedure for the optimal portfolio that maximizes the expected utility of terminal wealth. The accuracy of their model relied on the models for the price processes of derivative contracts.

The LSE's hedging problem of price and quantity risk under A Value-at-

[4] A straddle is a combination of a call and put at the same strike price. A straddle is used as a hedging instrument especially for protection against high volatility.

Risk (VaR) criteria has been considered by Woo et al. [WKH04], Wagner et al. [WSI03], and Kleindorfer and Li [KL05]. The VaR, which is defined as a maximum possible loss with $(1-\alpha)$ percent confidence, is a widely-used risk measure in practice which has become a standard tool in risk management. However, the optimization problems with the VaR risk measure are hard to solve analytically without very restrictive assumptions, especially when price and quantity risks are considered.

Woo et al. [WKH04] solved for a forward position q in order to minimize the expected procurement cost $PQ+(F-P)q$ subject to the VaR constraint where P, Q, and F are spot price, demand, and forward price, respectively. They solved the problem heuristically using a simple spreadsheet by setting possible hedge ratios first, and examining the risk exposure on total cost. Their normal distribution assumption on the monthly average spot price and load as well as the procurement cost simplified the calculation of the VaR measure.

More rigorous optimization was performed by Wagner et al. [WSI03] to determine the amounts of monthly forward contracts to be purchased for the upcoming several months. They considered an LSE that has to supply power at a fixed rate. They provided a simulation-based algorithm to solve the VaR-constrained problem, the problem of maximizing the expected hedged profit under the VaR constraint. However their method is inefficient because one has to evaluate VaR for all possible combinations on the number of different forward contracts.

To handle VaR analytically, the normality assumption is usually imposed on the hedged cash flow as in Ahn et al [ABRW99]. However, this normality assumption is not suitable for problems where the cash flow distribution is fat-tailed, like an LSEs's cash flow. Kleindorfer and Li [KL05] instead found a more relaxed assumption than normality while still maintaining the tractability of the normal distribution. Basically, when VaR is monotone in the variance, multi-period VaR-constrained problems were shown to be equivalent to mean-variance problems. Moreover, they solved the mean-variance problem that included various types of contracts including options over the planning horizon by transforming them into solvable quadratic programs.

Nasakkala and Keppo [NK05] also studied hedging of electricity cash flows with forward contracting strategies. It is basically a multi-period extension of McKinnon's problem [McK67] which determines the optimal hedge ratio in the presence of price and quantity risks. The quantity risk was modeled as a load estimate process, which represented the process of the estimates of load quantity at maturity. Static hedging strategies were considered because of concerns about transaction costs and illiquidity. When static hedging strategies are used, the agent faces at any point in time the decision of whether to hedge based on the current load estimate or wait for new information. They found the optimal hedging ratio and timing that minimizes the variance of the portfolio's cash-flow.

It is also worth mentioning a few papers on portfolio optimization for the supply side, in the electricity market. Producers, especially ones who own hydro plants, or those that sell load following fixed price contracts also face severe volumetric risk because their production capacities or supply obligation highly depends on weather condition such as precipitation and temperature. Their operational decision regarding when and how much to produce should be combined with a hedging strategy in the spot, forward and options market, but the difficulties arise because an operational decision for one period affects decisions for the later periods. Fleten et al. [FWZ99], Gussow [Gus01], Herzog [Her02], and Unger [Ung02] deal with such problems and solve them using multi-period stochastic dynamic programming.

3 Optimal Static Hedging in a Single-period Setting

In this section we reproduce, for completeness, results from [OOD06] upon which upon which we build the optimal timing extension described in the subsequent section. The problem is solved in two steps. First we solve for the payoff of the optimal hedging portfolio, and then we replicate that payoff function with a portfolio of standard instruments.

3.1 Finding the Optimal Hedge Payoff Function

Suppose that a hedging portfolio consisting of electricity derivatives is constructed at time 0 whose payoff at time 1, $x(p)$, is a function of the spot price p at time 1, is received at time 1. The hedging portfolio may also include money market accounts, allowing the LSE to finance hedging instruments through loans payable at time 1. Let $y(p, q)$ be the LSE's profit from serving the customers' demand q at the fixed retail rate r at time 1. Then, the hedged profit $Y(p, q, x(p))$ - total profit including the net payoffs of the hedging portfolio - is given by

$$Y(p,q,x(p)) = y(p,q) + x(p). \tag{1}$$

where

$$y(p,q) = (r-p)q.$$

The LSE's risk preference is characterized by a concave utility function U defined over the total profit $Y(\cdot)$ at time 1. LSE's beliefs on the realization of spot price p and load q are characterized by a joint probability function $f(p,q)$ for positive p and q, which is defined on the probability measure P. On the other hand, let Q be a risk-neutral probability measure based on which the hedging instruments are priced, and $g(p)$ be the probability density function of p under Q. Because the electricity market is incomplete, there may exist infinitely many risk-neutral probability measures. In this paper, it is assumed that a specific measure, Q, was picked according to some criteria.[5]

Then, the formulation of the optimal static hedging problem is as follows:

$$\begin{aligned} \max_{x(p)} \;\; & E\big[U[Y(p,q,x(p))]\big] \\ s.t. \;\; & E^Q[x(p)] = 0 \end{aligned} \tag{2}$$

where $E[\cdot]$ and $E^Q[\cdot]$ denote expectations under the probability measure P and Q, respectively. The constraint requires the manufacturing cost[6] of the portfolio to be zero under a constant risk-free rate. This zero-cost constraint implies that purchasing derivative contracts may be financed from selling other derivative contracts or through money market accounts. In other words, under the assumption that there is no limit on the possible amount of instruments to be purchased and money to be borrowed, the model finds a portfolio from which the LSE obtains the maximum expected utility over total profit.

The Lagrangian function for the above constrained optimization problem is given by

$$\begin{aligned} L(x(p)) &= E\big[U(Y(p,q,x(p)))\big] - \lambda E^Q[x(p)] \\ &= \int_{-\infty}^{\infty} E\big[U(Y)|p\big] f_p(p)dp - \lambda \int_{-\infty}^{\infty} x(p)g(p)dp \end{aligned}$$

[5]There are many proposed criteria to choose the risk-neutral measure in incomplete markets. See Xu [Xu04] for this subject.

[6]A derivative price is an expected value of the discounted payoff under the risk-neutral measure Q. We ignore here transaction costs.

with a Lagrange multiplier λ and the marginal density function $f_p(p)$ of p under P. Differentiating $L(x(p))$ with respect to $x(\cdot)$ results in

$$\frac{\partial L}{\partial x(p)} = E\Big[\frac{\partial Y}{\partial x} U'(Y)\Big| p\Big] f_p(p) - \lambda g(p) \tag{3}$$

by the Euler equation. Setting (3) to zero and substituting $\frac{\partial Y}{\partial x} = 1$ from (1) yields the first order condition for the optimal solution $x^*(p)$ as follows:

$$E\big[U'(Y(p, q, x^*(p)))\big| p\big] = \lambda^* \frac{g(p)}{f_p(p)} \tag{4}$$

Here, the value of λ^* should be the one that satisfies the zero-cost constraint (2).

Proposition 1 *For an agent who maximizes mean-variance expected utility of profit,*

$$E[U(Y)] = E[Y] - \frac{1}{2} aVar(Y),$$

the optimal solution $x^*(p)$ to problem (2) is given as

$$x^*(p) = \frac{1}{a}\Big(1 - \frac{\frac{g(p)}{f_p(p)}}{E^Q[\frac{g(p)}{f_p(p)}]}\Big) - E[y(p, q)|p] + E^Q[E[y(p, q)|p]] \frac{\frac{g(p)}{f_p(p)}}{E^Q[\frac{g(p)}{f_p(p)}]} \tag{5}$$

Moreover, suppose the joint distributions of p and q, is a bivariate log-normal distributions as follows:

$$\begin{array}{ll} \textit{Under } P: & \log p \sim N(m_1, s^2), \quad \log q \sim N(m_q, u_q^2), \quad Corr(\log p, \log q) = \phi \\ \textit{Under } Q: & \log p \sim N(m_2, s^2) \end{array}$$

Then,

$$x^*(p) = \frac{1}{a}(1 - B_1(p)) - B_2'(p) + B_3' B_1(p) \tag{6}$$

where

$$\begin{aligned} B_1(p) = \;& e^{-\frac{(m_1-m_2)(m_1-3m_2)}{2s^2}} p^{\frac{m_2-m_1}{s^2}} \\ B_2'(p) = \;& (r-p) e^{m_q + \phi\frac{u_q}{s}(\log p - m_1) + \frac{1}{2}u_q^2(1-\phi^2)} \\ B_3' = \;& r e^{m_q + \phi\frac{u_q}{s}(m_2-m_1) + \frac{1}{2}u_q^2(1-\phi^2) + \frac{1}{2}\phi^2\frac{u_q^2}{s^2}s^2} \\ & - e^{m_2 + m_q + \phi\frac{u_q}{s}(m_2-m_1) + \frac{1}{2}u_q^2(1-\phi^2) + \frac{1}{2}(\phi\frac{u_q}{s}+1)^2 s^2} \end{aligned}$$

Proof The proof is given in the Appendix.

Corollary 1 *Under the assumption of $P \equiv Q$, the optimal payoff function under the mean-variance expected utility becomes*

$$x^*(p) = E[y(p,q)] - E[y(p,q)|p] \tag{7}$$

Proof If $P \equiv Q$, then $\frac{g(p)}{f_P(p)} = 1$. Then, Eq. (5) reduces to Eq.(7). **QED**.

The assumption that $P \equiv Q$ was empirically justified by Audet et al. [AHKI04] and Koekebakker and Ollmar [KO05] for the Nordic electricity forward market. The first term $E[y(p,q)]$ in Eq. (7) is a constant, and the second term $E[y(p,q)|p]$ is the expected profit given the value of the spot price. The formula implies that the optimal payoff is one that levelizes the conditional expectation of hedged profit across spot prices p. This is because maximizing the mean-variance objective function given the zero-cost constraint and $P \equiv Q$ is the same as just minimizing a variance of hedged profits.[7] In fact, given the value of p, the variance of profit is zero after adding the optimal payoff in (7). This means that the optimal portfolio removes all the uncertainty in the profit that is correlated with price.

3.2 Replication of Exotic Payoffs

This section explores a way to construct a portfolio with exotic payoff $x(p)$, which was obtained in the last section.

Carr and Madan [CM01] showed that any twice continuously differentiable function $x(p)$ can be written in the following form:

$$x(p) = [x(s) - x'(s)s] + x'(s)p + \int_0^s x''(K)(K-p)^+ dK + \int_s^\infty x''(K)(p-K)^+ dK$$

for an arbitrary positive s.[8]

[7] This kind of hedging is also considered in [NK05]: mean-variance hedging reduces to variance minimization when the pricing measure equals to the physical measure because they consider only forward contracts, which have zero expected value before delivery.

[8] The simplest way of proving the formula is as follows: $\int_0^s x''(K)(K-p)^+ dK + \int_s^\infty x''(K)(p-K)^+ dK = \int_s^p x''(K)(p-K)dK = [x'(K)(p-K)]_s^p + \int_s^p x'(K)dK = -x'(s)(p-s) + x(p) - x(s)$; the first equality was obtained by considering the both cases of $p < s$ and $p \geq s$, and the second equality was from the integration by part.

This formula suggests a way of replicating the payoff function $x(p)$. Let F be the forward price for delivery at time 1. Evaluating the equation at $s = F$ and rearranging it gives

$$x(p) = x(F) \cdot 1 + x'(F)(p - F) + \int_0^F x''(K)(K-p)^+ dK + \int_F^\infty x''(K)(p-K)^+ dK. \quad (8)$$

Note that $1, (p-F), (K-p)^+$ and $(p-K)^+$ in the above expression represent payoffs at time 1 of a bond, forward contract, European put options, and European call options, respectively.

Therefore, an exact replication can be obtained from a long cash position of size $x(F)$, a long forward position of size $x'(F)$, long positions of size $x''(K)dK$ in puts struck at K, for a continuum of $K < F$, and long position of size $x''(K)dK$ in calls struck at K, for a continuum of $K > F$.

Note that unless the optimal payoff function is linear, the optimal strategy involves purchasing (or selling short) a spectrum of both call and put options with continuum of strike prices. This result proves that in order to hedge price and quantity risks together, LSEs should purchase a portfolio of options. The strike prices of call options effectively works as putting price caps on each incremental load.

In practice, electricity derivatives markets, as any derivatives markets, are incomplete. Consequently, the market does not offer options for the full continuum of strike prices, but typically only a small number of strike prices are offered. To implement the above replicating strategy using a discrete set of standard options contracts, discretization of strike prices is needed to approximate the optimal payoff function at each spot price realization. We provide here an approximate replication of an exotic payoff function using the existing Vanilla options so that the total payoff from those options is close to the exotic payoff.

Suppose there are put options with strike prices $K_1 < \cdots < K_n = F$ and call options with strike prices $F = K'_1 < \cdots < K'_m$ in the market. Letting $K_{n+1} = K_n$, $K_0 = 0$, $K'_0 = K'_1$, and $K'_{m+1} = \infty$, consider the following strategy, which consists of

a long cash position of size $x(F)$,

a long forward position of size $x'(F)$,

long positions of size $\frac{1}{2}(x'(K_{i+1}) - x'(K_{i-1}))$ in puts struck at K_i, $(i = 1, \cdots, n)$, long positions of size $\frac{1}{2}(x'(K'_{i+1}) - x'(K'_{i-1}))$ in calls struck at K'_i ($i = 1, \cdots, m$),

This strategy was obtained by the following approximations:
$\int_0^F x''(K)(K-p)^+ dK + \int_F^\infty x''(K)(p-K)^+ dK$
$= \sum_{i=0}^{n-1} \int_{K_i}^{K_{i+1}} x''(K)(K-p)^+ dK + \sum_{i=1}^{m} \int_{K'_i}^{K'_{i+1}} x''(K)(p-K)^+ dK$

$$\approx \sum_{i=0}^{n-1} \int_{\max(p,K_i)}^{\max(p,K_{i+1})} x''(K)dK \cdot \frac{1}{2}\{(K_i - p)^+ + (K_{i+1} - p)^+\}$$
$$+ \sum_{i=1}^{m} \int_{\min(p,K'_i)}^{\min(p,K'_{i+1})} x''(K)dK \cdot \frac{1}{2}\{(p - K'_i)^+ + (p - K'_{i+1})^+\}$$

$$\approx \sum_{i=0}^{n-1} \int_{K_i}^{K_{i+1}} x''(K)dK \cdot \frac{1}{2}\{(K_i - p)^+ + (K_{i+1} - p)^+\}$$
$$+ \sum_{i=1}^{m} \int_{K'_i}^{K'_{i+1}} x''(K)dK \cdot \frac{1}{2}\{(p - K'_i)^+ + (p - K'_{i+1})^+\}$$

$= \sum_{i=1}^{n} \int_{K_{i-1}}^{K_{i+1}} x''(K)dK \cdot \frac{1}{2}(K_i - p)^+ + \sum_{i=1}^{m} \int_{K'_{i-1}}^{K'_{i+1}} x''(K)dK \cdot \frac{1}{2}(p - K'_i)^+$.

As can be seen in the approximation, the error from the replicating strategy will be very close to zero if $x''(p)$ is a constant in each interval between two consecutive strike prices, and if p is realized very close to one of the strike prices. The error will be smaller if the intervals between two consecutive strike prices are smaller, especially for the interval within which there is a high probability that p will fall.

3.3 An Example

Consider a hypothetical LSE that is characterized by the following assumptions:

- Price is distributed lognormally with parameters $m_1 = 4$ and $s = 0.7$ in both the real-world and risk-neutral world: $\log p \sim N(4, 0.7^2)$ in P and Q. The expected value and the standard deviation of price p under this distribution is \$70/MWh and \$56/MWh.

- The LSE charges a flat retail rate $r = \$120/MWh$ to its customers.
- Load is lognormally distributed with parameter $m = 7.99$ and $u = 0.2$.

From Figure 1(a) one can see illustrations of the optimal payoff functions obtained under the mean-variance expected utility by varying correlations. When correlation is zero, the payoff function becomes linear, meaning that a derivative with a linear payoff is sufficient. However when there is positive correlation, the optimal payoff demonstrates nonlinearity, telling us that there's a need for derivatives other than forward contracts.

Figure 1(b) shows that the distributions of profits with price hedging[9] and with price & quantity hedging[10]. First, observe that both hedges reduce the variance of profit relative to the unhedged profit. Second, observe that price & quantity hedge cuts a left-tail of the distribution of profit after price hedge only. This implies that the LSE can reduce rare but detrimental events by hedging quantity risk. Moreover, the LSE can benefit from longer right-tail of the profit distribution after quantity hedging.

The replication strategy of the optimal payoff function in Figure 1(c) is shown in Figure 1. Assuming that options are available for strike prices of F (the forward price) and each increment and decrement of \$10 starting from F, Figure 1(c) shows the number of each of the contracts that should be purchased based on the discretization method developed in Section 3.2. The figure also shows that the forward contract covers slightly less than the expected demand (3000MWh), while call options are used to cover the incremental demand corresponding to high spot prices. Figure 1(d) confirms that the total payoff from our replication is very close to the optimal payoff function $x(p)$ that we want to replicate.

4 Timing of a Static Hedge in a Continuous-time Setting

Let T be the delivery period and maturing date of the hedging instruments. We will assume that all the hedging instruments for the delivery period T

[9]Price hedge here means that we add the optimal payoff function obtained under the assumption of no quantity risk. This is in fact equivalent to buying forward contracts for the average load quantity.

[10]Price and quantity hedge refers to the optimal payoff function that we obtained in this paper.

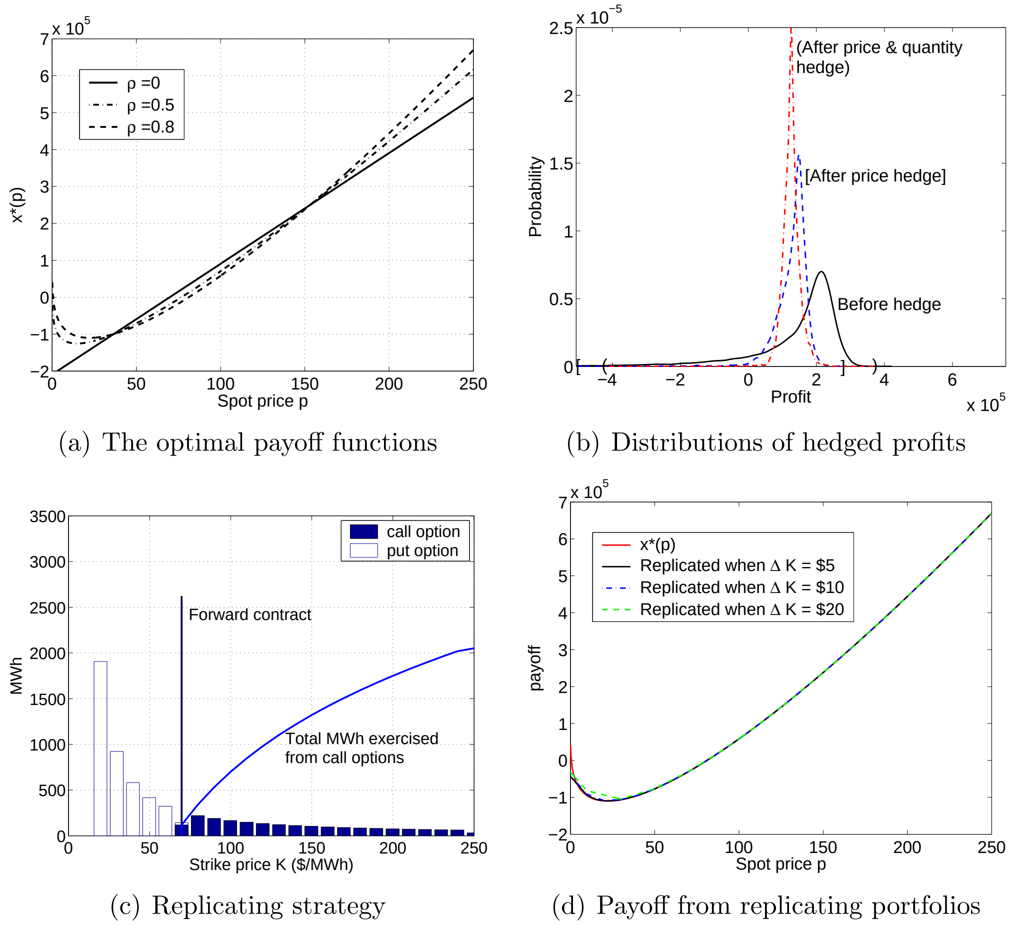


Figure 1: Hedging with mean-variance utility (minimizing variance) under price and load following bivariate lognormal distribution: $\log p \sim N(4, 0.7^2)$, $\log q \sim N(7.99, 0.2^2)$ with correlation coefficient $\rho = 0.8$. $(P = Q)$

are to be contracted at the same time τ. Contracting earlier reduces the risk by locking in the price of the contracts; while delaying the contracting enables more profitable hedging by exploiting more information that becomes available as we approach maturity. We will assume that the optimal hedging time is determined at time t_0 based on the information available at that time but the composition of the hedging portfolio is determined at hedging time based on the realized spot price. It should be noted that the optimal hedging time determined at t_0 might be no longer optimal at time $t_1 > t_0$ because more information on time T becomes available at time t_1, however we will assume that the contracting time is chosen irreversibly at time t_0.

4.1 Mathematical Formulation

Let $\{p_t\}_{t\in[0,T)}$ be a process of forward price for delivery at time T and $\{q_t\}_{t\in[0,T)}$ be a process for load estimate for period T calculated at time t. Assume the forward price and load estimate processes evolve as the following Ito processes:

$$dp_t = p_t\big(\mu_p(t)dt + \sigma_p(t)dB_t^1\big) \tag{9}$$

$$dq_t = q_t\big(\mu_q(t)dt + \sigma_{pq}(t)dB_t^1 + \sigma_q(t)dB_t^2\big) \tag{10}$$

where B_t^1 and B_t^2 are independent Wiener processes. Then, p_T and q_T denote the spot price and demand at time T.

The solution of the following problem is the best hedging timing determined at time t_0 for a mean-variance optimizer:

$$\max_{\tau\geq t_0} \; E_{t_0}[U((r-p_T)q_T + x_\tau(p_T))] \tag{11}$$

$$\text{s.t.} \quad x_\tau(p_T) = \arg\max_{x(p_T)} E_\tau[U((r-p_T)q_T + x(p_T))] \text{ s.t. } E_\tau^Q[x(p_T)] = 0 \tag{12}$$

$x_\tau(p_T)$ denotes the payoff from the optimal portfolio to be constructed at time τ. Thus, the formulation finds a time τ^*, hedging at that time maximizes the expected utility of the optimally hedged profit.

Throughout the section, it is assumed that the physical probability measure and risk-neutral probability measure are the same. It follows that from the zero-cost constraint, maximizing a mean-variance objective function is reduced to minimizing the variance of the hedged profit. The formula for $x_\tau(p_T)$ can be obtained from the results of the Section 3 (see the equation (7)). Thus, the problem becomes a single-variable unconstrained optimization problem that can be easily solved numerically.

4.2 Finding the Optimal Payoff Function at Contracting Time

Proposition 2 *Suppose* $\{p_t\}_{t\in[0,T]}$ *and* $\{q_t\}_{t\in[0,T]}$ *follow Ito processes given (14) and (15). Assuming* $P = Q$*, then* $x_\tau(p_T)$ *that solves (12) for a mean-variance utility function is obtained as follows:*

$$x^*_\tau(p_T) = B_\tau(p_T - r)p_T^{A_\tau}p_\tau^{-A_\tau}q_\tau + rC_\tau q_\tau - D_\tau p_\tau q_\tau \tag{13}$$

where

$$\begin{aligned}
A_\tau &= \frac{\int_\tau^T b_t d_t dt}{\int_\tau^T b_t^2 dt} \\
B_\tau &= \exp\Big(\int_\tau^T c_t dt - A_\tau \int_\tau^T a_t dt + \tfrac{1}{2}\int_\tau^T (d_t^2 + e_t^2)dt - \tfrac{1}{2}\frac{(\int_\tau^T b_t d_t dt)^2}{\int_\tau^T b_t^2 dt}\Big) \\
C_\tau &= e^{\int_\tau^T (c_t + \frac{1}{2}d_t^2 + \frac{1}{2}e_t^2)dt} \\
D_\tau &= e^{\int_\tau^T (a_t + c_t + \frac{1}{2}b_t^2 + \frac{1}{2}d_t^2 + \frac{1}{2}e_t^2 + b_t d_t)dt}
\end{aligned}$$

Proof The proof is given in the Appendix.

Equation (13) is the payoff of the optimal portfolio to be constructed when hedging at time τ. One can see that the optimal portfolio incorporates the information of the forward price and load estimate available at the hedging time τ.

4.3 Determining the Optimal Hedging Time

With the assumption $P = Q$ and the zero-cost constraint $E_\tau[x_\tau] = 0$, maximizing (11) reduces to minimizing

$$\Pi(\tau) \equiv Var((r - p_T)q_T + x^*_\tau(p_T)).$$

Given x^*_τ obtained in Section 4.2, the problem (11) is in fact an unconstrained optimization problem with a single decision variable in the interval $[0, T]$. Once Π is obtained as a function of τ, the problem is solvable numerically even though $\Pi(\tau)$ is neither convex or concave. This section is concluded with the calculation of $\Pi(\tau)$:

$$\Pi(\tau) = Var(x^*_\tau(p_T)) + 2cov((r - p_T)q_T, x^*_\tau(p_T)) + Var((r - p_T)q_T)$$

where

$$x^*_\tau(p_T) = B_\tau p_T^{A_\tau+1} p_\tau^{-A_\tau} q_\tau - rB_\tau p_T^{A_\tau} p_\tau^{-A_\tau} q_\tau + rC_\tau q_\tau - D_\tau p_\tau q_\tau$$

Each term of $\Pi(\tau)$ is calculated as a function of τ as follows: (For notational convenience, subscript τ for A_τ, B_τ, C_τ and D_τ is omitted.)

$$\begin{aligned}
Var(x^*_\tau(p_T)) &= E[x^*_\tau(p_T)^2] \\
&= B^2 E[p_T^{2A+2} p_\tau^{-2A} q_\tau^2] + r^2 B^2 E[p_T^{2A} p_\tau^{-2A} q_\tau^2] \\
&\quad + r^2 C^2 E[q_\tau^2] + D^2 E[p_\tau^2 q_\tau^2] - 2rB^2 E[p_T^{2A+1} p_\tau^{-2A} q_\tau^2] \\
&\quad + 2rBCE[p_T^{A+1} p_\tau^{-A} q_\tau^2] - 2BDE[p_T^{A+1} p_\tau^{-A+1} q_\tau^2] \\
&\quad - 2r^2 BCE[p_T^{A} p_\tau^{-A} q_\tau^2] + 2rBDE[p_T^{A} p_\tau^{-A+1} q_\tau^2] \\
&\quad - 2rCDE[p_\tau q_\tau^2] \\
cov((r-p_T)q_T, x^*_\tau(p_T)) &= E[(r-p_T)q_T x^*_\tau(p_T)] \\
&= rBE[p_T^{A+1} q_T p_\tau^{-A} q_\tau] - r^2 BE[p_T^{A} q_T p_\tau^{-A} q_\tau] \\
&\quad + r^2 CE[q_T q_\tau] - rDE[q_T p_\tau q_\tau] \\
&\quad - BE[p_T^{A+2} q_T p_\tau^{-A} q_\tau] + rBE[p_T^{A+1} q_T p_\tau^{-A} q_\tau] \\
&\quad - rCE[p_T q_T q_\tau] + DE[p_T q_T p_\tau q_\tau] \\
Var((r-p_T)q_T) &= E[(rq_T - p_T q_T)^2] - E[rq_T - p_T q_T]^2 \\
&= r^2 E[q_T^2] - 2rE[p_T q_T^2] + E[p_T^2 q_T^2] \\
&\quad - (rE[q_T] - E[p_T q_T])^2
\end{aligned}$$

The expectation terms were calculated using

$$\begin{aligned}
E[p_T^\alpha q_T^\beta p_\tau^\gamma q_\tau^\delta] = p_0^{\alpha+\gamma} q_0^{\beta+\delta} e^{\int_0^T (\alpha a_t + \beta c_t)dt} e^{\int_\tau^T (\frac{1}{2}(\alpha b_t + \beta d_t)^2 + \frac{1}{2}\beta^2 e_t^2)dt} \\
\cdot e^{\int_0^\tau (\gamma a_t + \delta c_t + \frac{1}{2}((\alpha+\gamma)b_t + (\beta+\delta)d_t)^2 + \frac{1}{2}(\beta+\delta)^2 e_t^2)dt}
\end{aligned}$$

from

$$\begin{aligned}
p_T^\alpha &= p_0^\alpha \exp(\textstyle\int_0^T \alpha a_t dt + \int_0^T \alpha b_t dB_t^1) \\
q_T^\beta &= q_0^\beta \exp(\textstyle\int_0^T \beta c_t dt + \int_0^T \beta d_t dB_t^1 + \int_0^T \beta e_t dB_t^2) \\
p_\tau^\gamma &= p_0^\gamma \exp(\textstyle\int_0^\tau \gamma a_t dt + \int_0^\tau \gamma b_t dB_t^1) \\
q_\tau^\delta &= q_0^\delta \exp(\textstyle\int_0^\tau \delta c_t dt + \int_0^\tau \delta d_t dB_t^1 + \int_0^\tau \delta e_t dB_t^2).
\end{aligned}$$

4.4 An Example

We now illustrate the optimal hedging timing with a concrete example. The example assumes that the maturity of the portfolio is one year from now. Base values of the parameters are set according to the empirical estimates of

Audet et al [AHKI04], which was also used by Näsäkkälä and Keppo [NK05]. Specifically, set

$$\mu_p(t) = 0, \quad \sigma_p(t) = e^{-\psi(T-t)}\sigma$$

where σ is the spot volatility and ψ is a mean-reversion rate of the spot price process, i.e., a rate at which forward volatility is discounted from the spot volatility. We also set $\mu_q(t) = 0$. In addition, $\sigma_{pq}(t)$ and $\sigma_q(t)$ are assumed as constants, so as to have a constant load volatility and correlation:

$$\sigma_L = \sqrt{\sigma_{pq}^2 + \sigma_q^2}, \qquad \phi = \frac{\sigma_{pq}}{\sigma_L}.$$

The resulting process is then

$$\frac{dp_t}{p_t} = e^{-\psi(T-t)}\sigma dB_t^1 \tag{14}$$

$$\frac{dq_t}{q_t} = \phi\sigma_L dB_t^1 + \sqrt{1-\phi^2}\sigma_L dB_t^2. \tag{15}$$

The forward price and load estimate for a month one year later is assumed to be 20Euro/MWh and 1000 MWh. The following table summarizes the base values of the parameters:

Parameter	T	r	p_0	q_0	ψ	σ	σ_L	ϕ
Value	1	40	20	1000	4.02	0.7	0.1	0.7

To study how the optimal hedging time is affected by various parameters, a sensitivity analysis of optimal hedging time with respect to parameter values is illustrated Figure 2.

Figure 2 (a) plots the optimal hedging time against the spot price volatility σ, and it shows that a higher spot volatility favors earlier hedging. Intuitively, a higher spot volatility increases uncertainties in the future information, which justifies locking in the price of hedging contracts earlier.

Figure 2 (b) plots the optimal hedging time against the load volatility σ_L. It shows that a higher volatility in the load estimate postpones the hedging time, confirming the intuition that the inaccuracy in the load estimate will delay the hedging so as to obtain more information.

Figure 2 (c) plots the optimal hedging time against the correlation between forward price and load estimate. It shows that a lower correlation

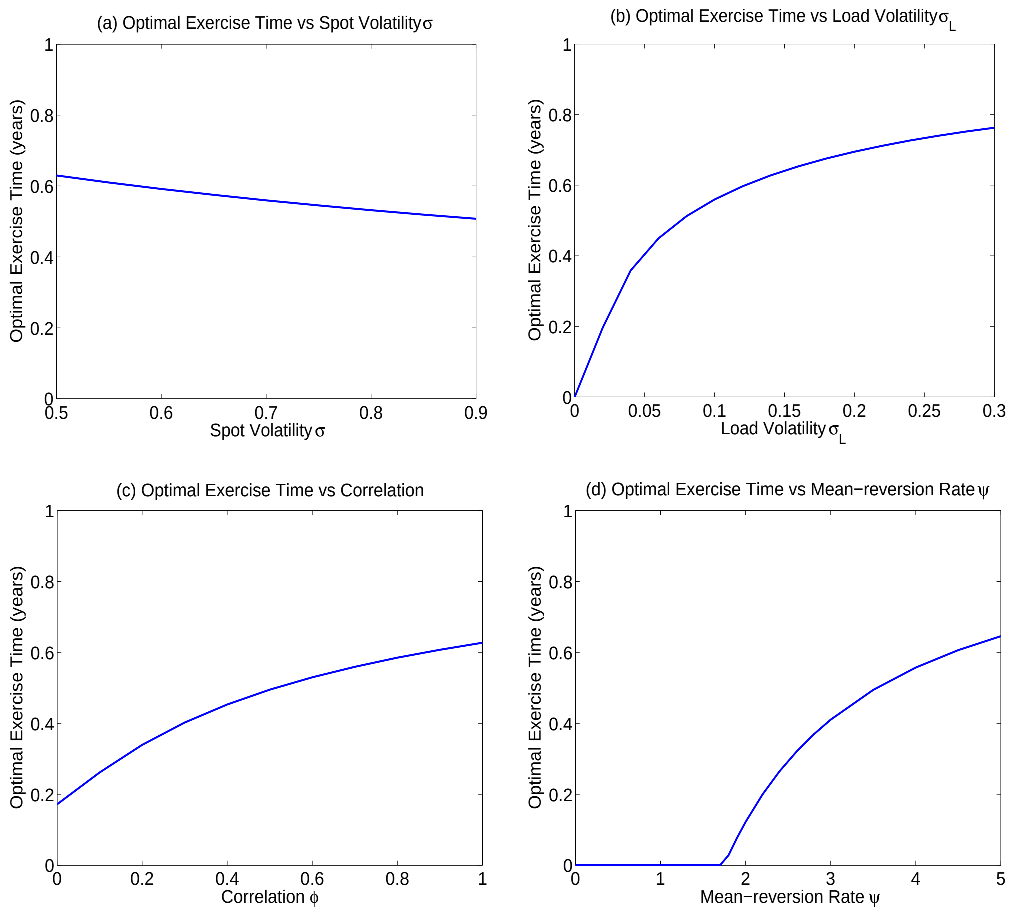


Figure 2: Optimal Hedging Time versus Other Parameter Values

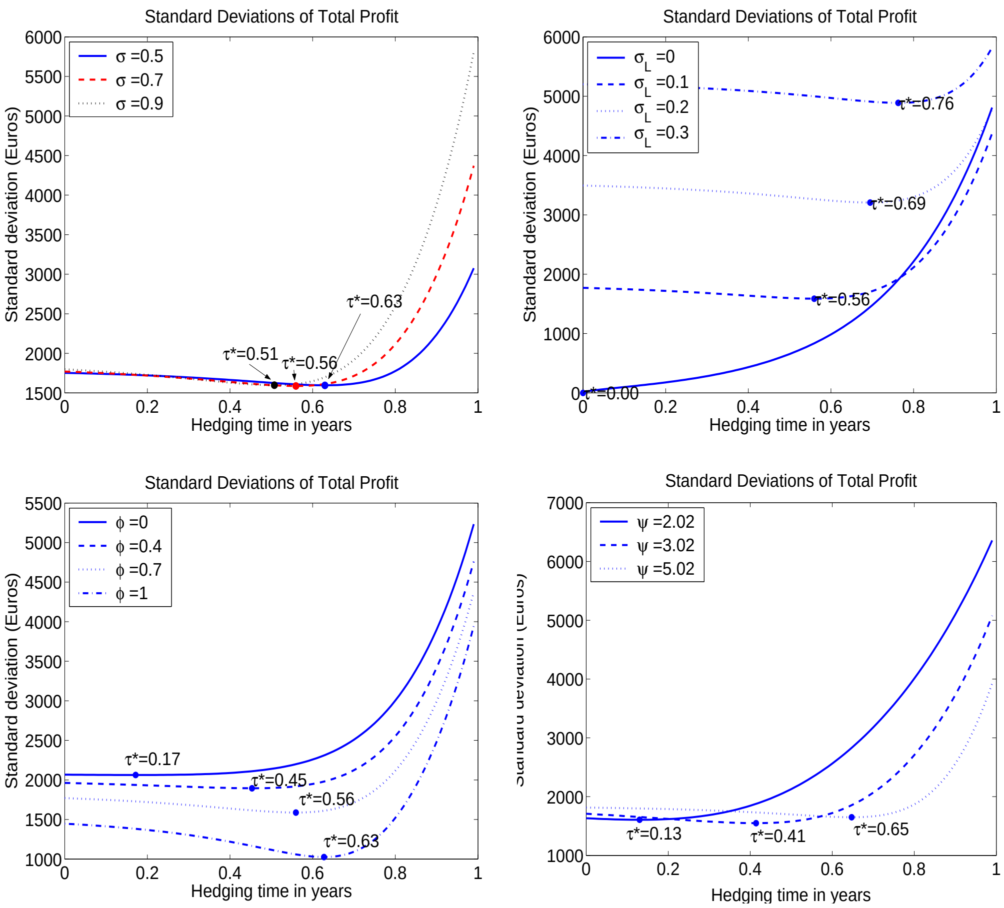


Figure 3: Standard Deviation of Hedged Profit Versus Hedging Times

makes earlier hedging more favorable which can be explained by the fact that high correlation reduces uncertainties in profit and thus delays hedging to take advantage of more information.

Figure 2 (d) plots the optimal hedging time against the mean-reversion rate of spot price. The figure shows that increase in mean-reversion rate of the spot price postpones the hedging time, since higher mean-reversion rate of spot price decreases the volatility of forward prices, so it will not be as risky to postpone the hedging time.

Figure 3 the variance of the optimally hedged profit as function of the hedging time. We note that hedging at time 0 versus the optimal time τ^* make little difference in the variance of hedged profit in most cases. How-

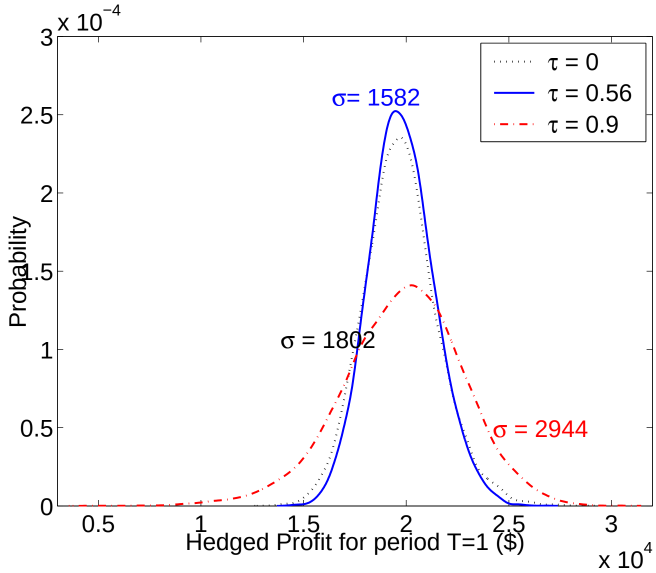


Figure 4: The distributions of profits when hedging at different times. The $\tau = 0.56$ is the distribution of the profits with hedging at the optimal hedging time, based on the information given at time 0.

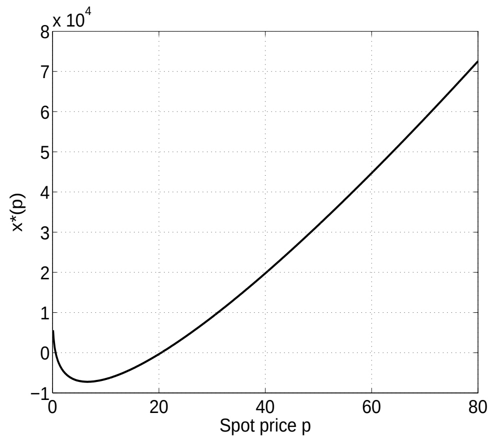
ever, the variance of profit increases rapidly if hedging is delayed beyond the optimal time.

Figure 3 also shows how the level of uncertainties changes with respect to the changes in σ, σ_L, ϕ, and ψ. The data displayed in the figure indicates that the profit uncertainty increases with the increases in spot and load volatility, and decreases with the mean-reversion rate and correlation coefficient.

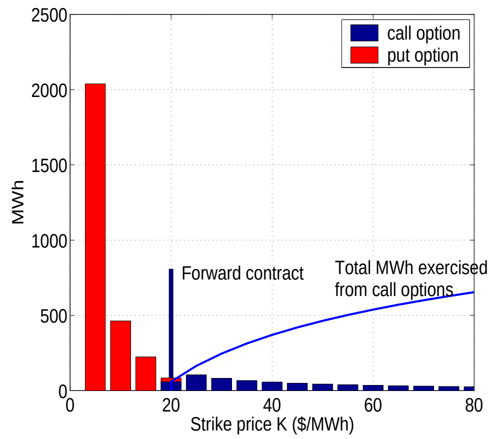
It is also noteworthy that hedging at the optimal time may not make any differences in the variance of hedged profit even for different parameters such as volatility and mean-reversion rate of spot price. In other words, the increased uncertainty from higher volatility in the forward price can be overcome by the optimal choice of hedging time.

Figure 4 compares the distributions of profits at delivery time when the hedging portfolio is purchased at time 0, at the optimal hedging time (0.56), and at time (0.9) close to delivery time. It also confirms that earlier hedging does not increase profit risk very much as compared to the optimal hedging time, but late hedging can have adverse consequences.

Finally, the optimal hedging strategy at time 0 under the base values



(a) The optimal payoff function



(b) The number of forward and options to purchase

Figure 5: The optimal payoff function and its replication when the hedging portfolio is constructed at time 0

of the parameters, is illustrated in Figure 5 which shows the optimal payoff function and its approximate replication (developed in Chapter 3) under the assumption that the hedging portfolio is constructed at time 0.

5 Conclusion

This paper developed a method of mitigating volumetric risk that load-serving entities (LSEs) and marketers of default service contract face in providing their customers' load following service at fixed or regulated prices while purchasing electricity or facing an opportunity cost at volatile wholesale prices. Exploiting the inherent positive correlation and multiplicative interaction between wholesale electricity spot price and demand volume, we developed a hedging strategy for the LSE's retail positions (which is in fact a short position on unknown volume of electricity) using electricity standard derivatives such as forwards, calls, and puts.

The optimal hedging strategy was determined based on expected utility maximization, which has been used in the hedging literature to deal with non-tradable risk. We derived an optimal payoff function that represent the payoff of the optimal costless exotic option as a function of price. We then showed how the optimal exotic option can be replicated using a portfolio of forward contracts and European options.

The examples demonstrated how call and put options can improve the hedging performance when quantity risk is present, compared to hedging with forward contracts alone. While at present the liquidity of electricity options is limited, the use of call options has been advocated by Oren [Ore05] and Chao and Wilson[CW04] in the electricity market design literature as a tool for resource adequacy, market power mitigation, and spot volatility reduction. These authors advocated capacity payments in the form of option premiums that will incent capacity investment, and ensure electricity supply at a predetermined strike price. The result of this paper contributes to better understanding of how options can be utilized in hedging the LSE's market risk, and hopefully increase their liquidity in the electricity market.

This paper extended previous work by considering the optimal timing of a hedging portfolio as well as the co-optimization of the portfolio mix taking account of the timing. For mean-variance expected utility, we solved for the optimal hedging time, under classical assumption regarding the stochastic processes governing forward price and load-estimate. The example showed that generally there is a critical time beyond which the uncertainty in profit increases sharply while the uncertainty remains relatively constant before this critical time.

Sensitivity analysis results indicate that the optimal hedging time gets closer to the delivery period if the positive correlation between the forward price and load-estimate is higher, and if the load-estimate volatility is higher. It is also observed that delaying the hedging time past the optimum time can be very risky, while the earlier hedging makes little difference as compared with hedging at the optimal time. This suggest that in practice one should err by hedging early rather than taking the chance of being too late.

The model presented in the paper determined the best hedging portfolio assuming that LSE has unlimited borrowing capability. In practice, credit limits can become an impeding factor in purchasing the optimal hedging portfolio. An LSE may not be able to borrow enough upfront money to finance the options contracts. Therefore, a credit limit constraint, which limits the amount of money that can be borrowed to construct the portfolio, needs to be considered in future extension of our model. A dynamic hedging strategy rather than the static approach adapted in this paper is likely to improve the hedging performance and should be considered in future extension of this work.

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Appendix

Proof of Proposition 1
It follows from $Var(Y) = E[Y^2] - E[Y]^2$ that

$$U(Y) \equiv Y - \frac{1}{2}a(Y^2 - E[Y]^2).$$

From $U'(Y) = 1 - aY$, the optimal condition (4) is as follows:

$$E[1 - aY^*|p] = \lambda^* \frac{g(p)}{f_p(p)}.$$

Equivalently,

$$f_p(p) - aE[Y^*|p]f_p(p) = \lambda^* g(p). \tag{16}$$

Integrating both sides with respect to p from $-\infty$ to ∞, we obtain $\lambda^* = 1 - aE[Y^*]$. By substituting λ^* and $Y^* = y(p, q) + x^*(p)$ into (16) gives

$$f_p(p) - a\Big(E[y(p, q)|p] + x^*(p)\Big)f_p(p) = g(p) - a\Big(E[y(p, q)] + E[x^*(p)]\Big)g(p).$$

By rearranging, we obtain

$$x^*(p) = \frac{1}{a} - \frac{1}{a}\frac{g(p)}{f_p(p)} + \Big(E[y(p, q)] + E[x^*(p)]\Big)\frac{g(p)}{f_p(p)} - E[y(p, q)|p] \tag{17}$$

To cancel out $E[x^*(p)]$ in the right-hand side, we take the expectation under Q to the both sides to obtain

$$0 = \frac{1}{a} - \frac{1}{a}E^Q\Big[\frac{g(p)}{f_p(p)}\Big] + \Big(E[y(p, q)] + E[x^*(p)]\Big)E^Q\Big[\frac{g(p)}{f_p(p)}\Big] - E^Q[E[y(p, q)|p]], \tag{18}$$

and subtract Eq.(18)$\times \frac{g(p)/f_p(p)}{E^Q[g(p)/f_p(p)]}$ from Eq.(17). This gives the final formula for the optimal payoff function under mean-variance utility as in (5).

Moreover, Eq. (17) is obtained by the following calculations:

$$\begin{aligned} B_1(p) &\equiv \frac{\frac{g(p)}{f_p(p)}}{E^Q\Big[\frac{g(p)}{f_p(p)}\Big]} = \exp\Big(\tfrac{m_2 - m_1}{s^2}\log p + \tfrac{m_1^2 - m_2^2}{2s^2} - \tfrac{(m_1 - m_2)^2}{s^2}\Big) \\ &= e^{-\frac{(m_1 - m_2)(m_1 - 3m_2)}{2s^2}} p^{\frac{m_2 - m_1}{s^2}} \\ B_2'(p) &\equiv E[y(p, q)|p] = E[(r - p)q|p] = (r - p)e^{m_q + \phi\frac{u_q}{s}(\log p - m_1) + \frac{1}{2}u_q^2(1 - \phi^2)} \end{aligned}$$

since $\log q|p \sim N(m_q + \phi\frac{u_q}{s}(\log p - m_1), u_q^2(1-\phi^2))$, and

$$B_3' \equiv E^Q[E[y(p,q)|p]] = re^{m_q+\phi\frac{u_q}{s}(m_2-m_1)+\frac{1}{2}u_q^2(1-\phi^2)+\frac{1}{2}\phi^2\frac{u_q^2}{s^2}s^2} - e^{m_2+m_q+\phi\frac{u_q}{s}(m_2-m_1)+\frac{1}{2}u_q^2(1-\phi^2)+\frac{1}{2}(\phi\frac{u_q}{s}+1)^2 s^2} \quad (19)$$

since $\log q|p \sim N(m_q + \phi\frac{u_q}{s}(\log p - m_1), u_q^2(1-\phi^2))$. **QED**.

Proof of Proposition 2
Recall that, under $P = Q$, the optimal payoff function for the mean-variance optimizer was given by the following formula (see (7)):

$$x^*(p) = E[y(p,q)] - E[y(p,q)|p]$$

where $y(p,q) = (r - p_T)q_T$. The payoff $x_\tau(p_T)$ at (12) is then obtained by taking conditional expectations on time τ, instead of at time 0:

$$x_\tau^*(p_T) = E_\tau[(r - p_T)q_T] - (r - p_T)E_\tau[q_T|p_T]. \quad (20)$$

Given equations (14) and (15), Ito's formula obtains p_T and q_T as follows:

$$p_T = p_\tau \exp\{\int_\tau^T a_t dt + \int_\tau^T b_t dB_t^1\}$$
$$q_T = q_\tau \exp\{\int_\tau^T c_t dt + \int_\tau^T d_t dB_t^1 + \int_\tau^T e_t dB_t^2\}$$

where $a_t = \mu_p(t) - \frac{1}{2}\sigma_p^2(t)$, $b_t = \sigma_p(t)$, $c_t = \mu_q(t) - \frac{1}{2}\sigma_{pq}^2(t) - \frac{1}{2}\sigma_q^2(t)$, $d_t = \sigma_{pq}(t)$, and $e_t = \sigma_q(t)$. It follows that p_T and q_T conditional on time τ follows a bivariate log-normal distribution: $(\log p_T, \log q_T)|_\tau \sim N(m_1, m_q, s^2, u_q^2, \phi)$ where

$$\begin{aligned}
m_1 &= E_\tau[\log p_T] = \log p_\tau + \int_\tau^T a_t dt \\
m_q &= E_\tau[\log q_T] = \log q_\tau + \int_\tau^T c_t dt \\
s^2 &= Var_\tau(\log p_T) = \int_\tau^T b_t^2 dt \\
u_q^2 &= Var_\tau(\log q_T) = \int_\tau^T (d_t^2 + e_t^2) dt \\
\phi &= Corr_\tau(\log p_T, \log q_T) = \frac{E_\tau[\log p_T \log q_T] - E_\tau[\log p_T]E_\tau[\log q_T]}{s \cdot u_q} \\
&= \frac{(\log p_\tau + \int_\tau^T a_t dt)(\log q_\tau + \int_\tau^T c_t dt) + \int_\tau^T b_t d_t dt - (\log p_\tau + \int_\tau^T a_t dt)(\log q_\tau + \int_\tau^T c_t dt)}{s \cdot u_q} \\
&= \frac{\int_\tau^T b_t d_t dt}{s \cdot u_q}
\end{aligned}$$

Equation (20) is then calculated to give the following function:

$$
\begin{aligned}
x_\tau^*(p_T) = (p_T - r)\exp\left(m_q + \phi\frac{u_q}{s}(\log p_T - m_1) + \frac{1}{2}u_q^2(1-\phi^2)\right) \\
+ r\exp\left(m_q + \frac{1}{2}u_q^2\right) - \exp\left(m_1 + m_q + \frac{1}{2}(s^2 + u_q^2) + \phi s u_q\right) \quad (21)
\end{aligned}
$$

Equation (13) is obtained by substituting the parameters m_1, m_q, s, u_q, and ϕ into equation (21). **QED.**