Identification of Market Power in Large-Scale Electric Energy Markets

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Abstract

Market power potential is a serious concern for efficient and competitive operation of centrally-dispatched electricity markets. Traditional measures for market power ignore underlying physical characteristics of the electric grid that may be exploited for local advantage. In our prior work we have proposed a revenue sensitivity-based approach for identifying market participants with market power potential, and we demonstrated detailed cases using a 30-bus system[1][2][3]. In this paper we address computational challenges for scaling our method to large systems, and we present practical extensions to a portion of our work that enables the evaluation of very large, RTO-scale electric power grids.

I. Introduction

In a competitive environment with sufficient capacity to adequately meet demand and reserve requirements, a centrally-dispatched electricity market provides a transparent and efficient means for economic operation of the grid. When certain suppliers possess some advantage over other suppliers, then competition is jeopardized and the benefits of an electricity market may not be realized.

This issue of supplier advantage versus competition can be viewed in terms of *substitutability*. If the market is competitive, then the electricity supplied by one generator is easily substituted by that of another generator at, or very near, the market price. If a generator were to attempt to manipulate the market to its advantage, for example by maintaining dispatch while raising prices, they would not be able to do so. Their competitors would provide energy to substitute for the profit-ambitious supplier's higher-priced energy. Conversely, if a supplier, or small group of suppliers do possess some locational advantage that limits their competitors' ability to provide substitute power, then the profit-ambitious suppliers may be able to simultaneously maintain constant dispatch while increasing prices, and thereby increase revenues and profits.

Taking into account network limitations in the electric grid, such examples of local market power are possible, and expected, in so-called "load pockets." These are regions in which generation supply to meet demand is limited by network constraints (transmission capacity, voltage, or operational reliability constraints). Incremental increases in demand must be met by a small subset of suppliers, or even a single supplier, effectively reducing or eliminating the measure of substitutability.

Traditional metrics to help identify market power such as the HHI, pivotal supplier index, and residual supplier index (RSI)[6] when applied to the entire network do not completely address the issue of load pockets.(See [7] for a discussion of this issue.) Certainly if these metrics identify non-competitive operation, there should be immediate concern. However, if these metrics do not identify non-competitive operation, one should still be concerned about the possibility of local market power in load pockets. Of course these metrics can and sometimes are applied to smaller portions of the network. For example, in [5] the HHI index is applied to clusters of buses with similar prices within a network. But these metrics lack a representation for the electricity grid that can naturally distinguish cases of local market power.

In our approach, presented in previous work, we examine matrices of revenue-price sensitivity and dispatch-price sensitivity [1][2][3]. The former is used to identify suppliers with the ability to simultaneously raise prices and revenues, indicating market power potential. The latter can be directly used to identify load pockets in which suppliers may have locational advantage. The matrices are related through a simple formula. Importantly, both matrices are derived from a detailed model of an optimal dispatch program that includes a representation for the physical network and its limiting constraints. We demonstrated the potential for market power manipulation in a small system (30-bus, 6-supplier) both in theory and through experiment. Using the revenue and dispatch sensitivity matrices we were able to clearly identify the suppliers with joint market power that were observed in the experiment.

The mathematics used to develop the relevant matrices is admittedly sophisticated, and a complete search through the structure of the matrices to identify the suppliers with market power is fundamentally combinatorial. Unfortunately, we find that the computations required to perform these two tasks do not scale well to large systems of practical interest. We need to find ways to improve the analysis.

Specifically, given a matrix of revenue-price sensitivities, we need to search over all combinations of price perturbations that result in a revenue increase for candidates for market power. Or, in terms of load pocket identification, we need to group those suppliers which can raise prices with negligible impact on dispatch.

In [8] the authors simplified the search over the matrices by limiting the number of generator combinations that need to be considered, and assuming uniform increases in price for generators with market power. Even with these simplifications, the resulting combinatorial analyses will be prohibitive for large systems, and we are also interested in nonuniform price increases that may result in market power potential. In [3] we explored a spectral approach for identifying load pockets. While effective and allowing for nonuniform prices increases, it likewise is difficult to adapt to very large systems.

While we continue efforts to improve the efficiencies of such searches over the sensitivity matrices, we present here an intermediate screening process that allows the identification of suppliers with both market power potential and who are exercising this ability. This is accomplished through an evaluation of the geographical distribution of Locational Marginal Prices (LMPs). LMPs in the network separate when there is some impediment to trade between locations (losses, transmission congestion, voltage constraints, etc.). By clustering buses by LMP we screen for groups of generators that enjoy a privileged position in the network. These screened groupings are tested using an approximation of the sensitivity matrix [9][10] to confirm that the suppliers have market power ability.

While this approach is practical, it is essential to mention what we give up for these large-scale calculations. The clustering algorithm we employ is based on differences in locational marginal prices. It will allow us to identify suppliers that may already be exercising some market power to their advantage, but it will not identify those generators with market power potential that is not being exploited.

In the end, based on findings using this screening method, we test the hypothesis that certain suppliers share market power by performing detailed optimal power flows. If these generators can simultaneously raise prices and revenues, then they enjoy a locational advantage and potential for market power.

In the remaining sections of this paper we review mathematical approaches for calculating the sensitivity matrix, and outline the LMP-based clustering algorithm. Then we apply the clustering algorithm on a large model of the Western Interconnect.

II. Sensitivity Matrix Calculations

In this section we provide background on methods for calculating the matrix of dispatch-price sensitivities. This matrix, denoted by M below, relates how incremental changes in prices will result in changes in dispatch,

$$\Delta g = M\Delta\lambda \tag{1}$$

where $\Delta\lambda$ and Δg represent the changes in price and generation dispatch respectively.

Since generator revenue is equal to the product of price and dispatch, the matrix of revenue-price sensitivities, denoted by N, is simply related to M by

$$\begin{aligned}\Delta r &= \left[diag(\lambda)M + diag(g)\right]\Delta\lambda \\ &= N\Delta\lambda\end{aligned} \tag{2}$$

where $diag(\lambda)$ and $diag(g)$ represent diagonal matrices of generator nodal prices and generator dispatches respectively.

Before proceeding with a description of how one may compute the M sensitivity matrix we need to be clear about what we mean by dispatch-price sensitivity. In electricity markets, generator offers are typically submitted in discrete blocks, and the uniform clearing price will often differ from the block offer price at the generator location. The generator block offer curve is essentially a discontinuous marginal costs curve that is not amenable to direct sensitivity analysis. To obtain a meaningful incremental model, we substitute the generator nodal clearing price for the generator offer price. This is sensible in our incremental model because we want to know the effect on dispatch when a generator incrementally increases its offer over the LMP. By construction, an OPF run with these substituted offers will give the same result as the original OPF. By starting with the solution of an optimal power flow and replacing the generator offers with LMPs we essentially make all the generators marginal units, and then calculate the sensitivity of the dispatch to changes in price.

This optimization problem takes the following form:

$$\min \lambda^T g \tag{3}$$

subject to the constraints

$$f(z,y) = \begin{bmatrix} f_1(g,y) \\ f_2(y) \end{bmatrix} \tag{4}$$

where $f_1(g,y)$ contains the network active power constraints at the generator buses, $f_2(y)$ contains the rest of binding constraints (other power flow constraints, transmission capacity constraints, voltage constraints), and y represents all remaining relevant variables in an OPF. We omit the details here for brevity, but in [1] it is shown that the M sensitivity matrix can be derived from the linearized first-order conditions of the Lagrangian for this optimization problem. The exact M matrix takes the form

$$M = \frac{\partial f_1}{\partial y} \quad H^{-1} \frac{\partial f_2^T}{\partial y} \quad \frac{\partial f_2}{\partial y} H^{-1} \frac{\partial f_2^T}{\partial y} \quad ^{-1} \frac{\partial f_2}{\partial y} H^{-1} - H^{-1} \quad \frac{\partial f_1^T}{\partial y} \tag{5}$$

where H is a Hessian matrix associated with constraints f_1 and variables y. This matrix is difficult to obtain in practice since commercial power flow programs capable of solving OPFs for the large systems of ultimate interest do not do this calculation for us, or provide a means to extract the subcomponents in this equation. These programs could be adjusted to do so, and this may be beneficial to RTO market monitors.

In this work we apply an approximate method for estimating this matrix [9][10] that is practical for our research purposes. The estimate of the sensitivity matrix M is based on a simplified model of the power network that focuses on the key variables and dominant effects while greatly reducing the complexity of the representation. The key variables are generator dispatch and price, and the dominant phenomena are conservation of energy (of course!) and transmission congestion. An optimization problem with these features include the original minimum costs function

$$\min \lambda^T g \qquad (6)$$

subject to the conservation of energy constraint

$$1^T g = P_{Loss}(g) + P_D \qquad (7)$$

where P_D is the total system demand and $P_{Loss}(g)$ is a network loss function expressed in terms of power injections, and binding transmission constraints

$$Wg = P_{Max} \qquad (8)$$

where P_{Max} is a vector of maximum branch flow capacities for the constrained lines and W is a matrix of "shift factors" that express branch flow in terms of generator power injections. Note that the initial solution to the OPF provides the binding constraints, nominal dispatches and prices.

In [9][10] it is shown that the estimate for the dispatch-price sensitivity matrix is given by

$$M = \frac{B^{-1}}{\mu^*}\left[V^T\left(VB^{-1}V^T\right)^{-1}VB^{-1} - I\right] \qquad (9)$$

where

$$V = \begin{bmatrix} \lambda^T \\ W \end{bmatrix}$$

and B is a matrix of "B-factors" associated with the loss function $P_{Loss}(g)$ (see Appendix), W is the matrix of shift factors associated with binding transmission constraints, λ is the vector of prices, and μ^* is the shadow price associated with constraint .

The representation (9) of this estimate is not simple, but it has an advantage over the complex representation given by (5): it is easily calculated if one assumes a form for the loss function and can obtain the loss function "B-factors." We defer a specific discussion on how we compute these factors for our examples in this paper to the appendix. The more accurate the calculation for these factors, the better the approximation of the matrix will be. The only element of the representation that is not readily available or can be easily calculated is the value of μ^*. Since this is a scalar, however, it is not needed for analysis; any scaled version of M will allow us to identify load pockets.

III. A Clustering Algorithm

Using the dispatch-price sensitivity matrix, we now proceed to try to identify generators located in load pockets that may have some local advantage of limited substitutability. Specifically we seek combinations of generator price increases that result in no change in dispatch:

$$M\Delta\lambda = 0 \qquad (10)$$

(or equivalently $\mu^* M\Delta\lambda = 0$ since μ^* is a scalar). If a generator or a small group of generators can simultaneously raise prices without changing system dispatch, then they can increase their own revenues and have market power potential.

There are two approaches to find vectors, $\Delta\lambda$, that satisfy (10): a combinatorial approach over all elements of the vector, and a spectral approach that notes that a vector that satisfies (10) must lie in the null space of M. Given the form of (10) the latter approach seems direct and promising, so it is important to point out the computational challenges that may not obvious at first glance. In practice it is too restrictive to require equality in (10); we need to allow

$$\left|M\Delta\lambda\right| < \varepsilon \qquad (11)$$

where ε is some small limiting threshold for $|\Delta\lambda|=1$. Vectors that satisfy (11) can be computed using eigenanalysis, but these alone do not completely solve our problem. The eigenvectors will constitute a basis of vectors from which any linear combination will satisfy (11). That is where the difficulty lies, we need to search over the space of possible vectors for those with a very specific form – mostly zero entries and strictly positive nonzero entries. This is not impossible, but it does pose challenges.

Alternatively, if we modify our goal to seek suppliers with both market power potential and who are exploiting this ability, a simple clustering algorithm is practical. The intuition for this approach comes from the information that may be inferred from locational marginal prices. LMPs differ in the network when there is some impediment to the transfer of energy, such as resistances in transmission lines, and transmission congestion from capacity and voltage limits. A grouping of connected buses with similar LMPs suggests that there are no impediments to energy transfer among them, and any generators within this group will be substitutable. However, if there is significant variation between generator LMPs, then they will tend to not be substitutable.

Consider the groupings shown pictorially in Figure 1. There is transmission congestion that prevents incremental energy transfer between them. Within each group, energy transfer is unimpeded. If there is only a single supplier, or a small group of suppliers in any group, they have some amount of market power. In Figure 1, if there only a few suppliers in Group C, then they will have the ability to simultaneous raise prices and revenues, essentially without bound. If there are a limited number of suppliers in Group B then they will be able to raise price, up to a point. When the prices become comparable to those in Group C, the transmission congestion may be relieved and the number of substitutable generators will include those in both groups. Similarly, suppliers in Group A may be able to raise prices until they have to compete with the generators in Group B.

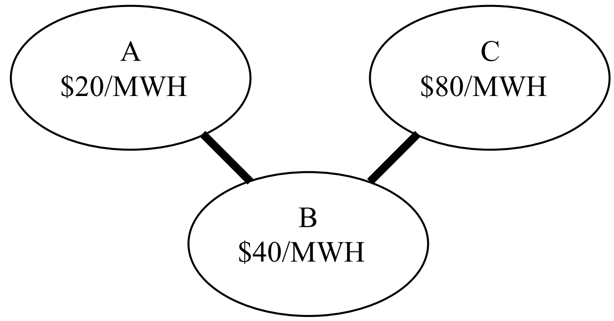

Figure 1. Three groups of buses with congestion between them.

We propose to cluster buses into groups based on LMP. The algorithm is simple:

1. Identify those branches in the network connecting buses with LMP differences exceeding a user-specified threshold. Keep these branches.
2. Remove the remaining branches for which the price difference is lower than the threshold by merging the terminal nodes. A sequential process of merging nodes forms the clusters.

The threshold in step 2 is important. In practice we must allow for some variation in LMPs within a group, but we do not want to miss clear cases of market power exploitation. Too low of a threshold will erroneously make distinctions that only arise due to expected small prices differences that always occur in a system with losses, and does not indicate uncompetitive behavior. Too high of a threshold will fail to identify instances of market power exploitation. Also, for the operation of the clustering algorithm it also presumed that threshold serves to separate clusters using the retained branches. One needs to check that these branches do connect different clusters after the procedure is complete. (This is a trivial check.)

Once we have performed the clustering, we examine those groups with only a small number of units or suppliers. These become candidates for further investigation, initially using the *M* sensitivity matrix, and ultimately, a detailed run of an OPF. Knowing which generators we are interested in allows us to directly examine the sensitivity matrix to confirm these generators are in a load pocket and have market power potential. We can indisputably confirm the generators have market power by running an OPF with increased prices and reexamine their dispatch and revenues. We do this for a large example system in the next section.

To be clear, this approach uses different information than that contained in the sensitivity matrix. The screening based on LMP clusters will not identify market power potential that is not evident in price differences. There may be other combinations of suppliers with market power potential that won't be identified in this manner.

IV. Western Interconnect Example

In this section we apply our clustering algorithm to a detailed model of the Western Interconnect, to identify load pockets with market power potential. The model has 13,374 buses, 16,000 lines, 105 GW of load, and 117 GW of on-line capacity. The model covers the Western United States and portions of Canada and Mexico. We study a hypothetical condition in which the Western United States portion is dispatched as a single electricity market. (Presently only California operates a central market, and only a balancing market.) This portion of the network for which we calculate LMPs and perform our clustering contains 10,841 buses, and has 88 GW of load and 107 GW of on-line capacity. We use the Powerworld Simulator optimal power flow to calculate dispatch and LMPs.

We apply a threshold of \$20/MWH in our clustering algorithm. This number is large enough to distinguish between congested groups and small enough to catch market power exploitation before it reaches an extreme level.

The clustering algorithm results in 70 groups. The largest covers most of the North and Central regions and contains 7318 buses and 860 generators. This large group has an average bus LMP of \$23.6/MWH. Including this group, there are 8 clusters with more than 100 buses. The 62 remaining clusters have fewer than 100 buses.

In our screening process we examine the clusters for groups that have four or fewer units with some remaining capacity (with which to be substitutable), and which are not at their minimum capacities. (If they are all at minimum dispatch, raising prices will neither change dispatch nor LMPs.) This screens four clusters with possible local market power. Some of their relevant information is provided in Table 1.

Table 1. List of clusters to be screen for market power.

cluster	18	26	30	32
LMP	63	17	87	50
Load	213	0	73	29
Buses	4	3	6	3
units	3	4	3	2

We show one of the representative clusters in detail to describe here in the paper. (All of them prove to have market power.) Cluster 17 is easy to depict – having only 4 buses – and contains a respectable amount of load. A local diagram of this load pocket is shown in Figure 2.

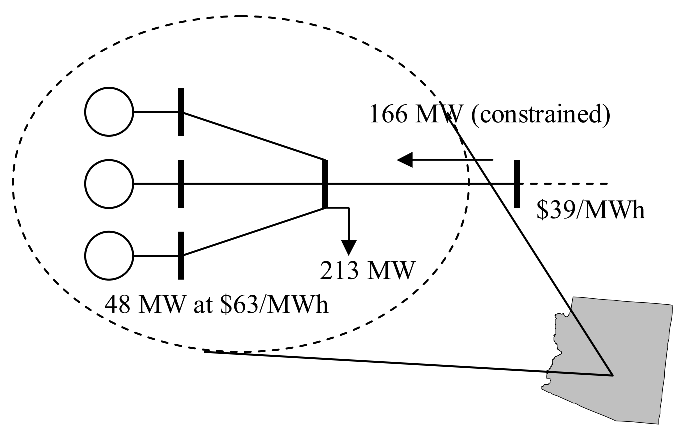

Figure 2. Cluster 17 Load Pocket.

From the diagram it is obvious that this is a load pocket. The load of 213 MW is met in part by 166 MW from outside the pocket through a capacity-constrained line. The remaining power is supplied from three local generators. Acting together, explicitly (if they have a common owner) or implicitly, they have the ability to raise prices without bound, with no effect on dispatch.

To confirm this in our mathematical description we examine the corresponding columns of the M sensitivity matrix to determine if a strictly positive linear combination of these columns will equal zero. This would confirm that it is possible to raise prices without changing dispatch.

For this example we compute the M matrix, or more precisely, $\mu^* M$, if the multiplier μ^* is not known. There are 910 active generators, and the dispatch-price sensitivity matrix is a completely full 910 by 910 matrix. First we extract the 3 by 3 submatrix corresponding to the three generators in Cluster 17:

–379.4	349.4	29.98
349.4	–379.3	29.89
29.98	29.89	–59.87

The negative diagonal entries suggest that any generator acting alone will decrease its market share. Acting together however, they can raise prices without changing dispatches. The eigenvalues for this submatrix are -728.8, -89.81 and -0.0124. The eigenvector corresponding to the (near) zero eigenvalue is

$$[0.5775 \quad 0.5775 \quad 0.5771]^T$$

A change in prices aligned with this eigenvector will result in no change in dispatch for these generators. Furthermore, a full check of the entire matrix M shows that such a change in prices will have little effect on the dispatch anywhere in the system. These generators have the ability to raise prices without changing dispatch, and hence enjoy market power potential. Since their prices are significantly higher than the neighboring portion of the system, there could be concern that market power is being exploited. Of course it is possible that the three units are all expensive peaking units and that the prices are justifiable. One should consider this (possibly automated) process as flagging units for further investigation.

The ultimate test for confirming market power ability is to run an OPF of the entire system with increased offer prices at the generators identified as having market power potential. In the case of the load pocket associated with Cluster 17, we find that these generators can raise the prices arbitrarily with no change in net dispatch among the three generators. They have unlimited market power in this location.

From the configuration shown in Figure 2, it is obvious that Cluster 17 is a load pocket in which the generators will have joint market power. There is a single constrained line into this area. Of the four clusters listed in Table 1, three are of this type; they are isolated by a single constrained line. The remaining cluster, Cluster 30, is connected to the rest of the system through two lines, only one of which is constrained. The eigenvalues of submatrix corresponding to the generator (buses) are –90.15 and –7.040.

At first glance it would appear that these four generators (located at two buses) in group 30 do not have market power potential. Perturbation studies with the OPF suggest the opposite. The generators in this load pocket can raise their prices without affecting dispatch. To resolve this seeming inconsistency we examined the M matrix to determine which generators should be substitutable for those in group 30 and we identify two generators located in a different cluster. Closer examination of the original OPF solution shows that these units are dispatched and operating at their minimum generation limits. In a practical sense, since their LMPs are lower than their offer prices, these generators are not substitutable for other sources, at least not in an incremental sense (and perhaps should not be included in the *M* matrix).

As we discussed earlier, some suppliers may enjoy limited market power having the ability to raise price until they must compete with other higher-priced suppliers. Since our threshold is set to $20/MWh, the suppliers in each load pocket should be able to raise their prices by this amount before reaching the price levels of the possible competitors. Repeated OPFs confirm that this is true for all four of the load pockets we identify using the clustering algorithm.

V. Discussion and Conclusions

In this paper we report on our recent activities to extend our work on market power monitoring to address practical problems associated with large systems. Our approach to detecting the potential for market power relies on the calculation and analysis of sensitivity matrices that relate price to dispatch and revenue. Brute force analysis of the matrices is combinatorial in nature because it is necessary to consider all combinations of suppliers for market power potential. The spectral approach we presented in [3] is promising and we continue to seek a way to apply the ideas to large systems.

Here, we consider an alternative approach to screen for supplier combinations that may have market power potential. Using a clustering algorithm that exploits information in LMP distributions, we identify candidates for further investigation that will use the sensitivity matrix and apply subsequent OPF perturbation analysis. To be clear, the information garnered from the LMP distribution is not the same information in the sensitivity matrix, and it can only detect load pockets and potential for market power that is evident from prices differences. It will not be possible, for example, to detect market power potential that does not result in a pattern of price differences in the network. Nevertheless, it provides another method for examining the network for some instances of market power; a topic of fundamental importance. This interim approach complements the commonly-used concentration-based methods mentioned in the Introduction.

In this paper we use the approximation for the sensitivity matrix found in [9][10] which requires knowledge of the system B-factors. In the appendix we describe how we calculated the factors for this study. The Appendix focuses on a particular load bus based distributed slack that provides a full rank B-factor matrix with entries for all the generators. This particular form is desirable because of the matrix inversions required in Equation (9). Otherwise we used a textbook approach for the approximation and it does not account for a shift in operating conditions. For a more accurate estimate we recommend calculating the factors using a linearization about the specified OPF operating point, and using a distributed slack similar to that presented here.

Our application to a large-scale model is encouraging. It demonstrates that the calculations are practical. In this study we limited our investigation to clusters containing no more than 4 generating units to focus on pockets with few suppliers. With more complete information, it would be better to focus on regions with few *owners* as this would be a more accurate description of "few suppliers."

Analysis of cluster 30 revealed that we may have to reconsider how we represent generators operating at minimum dispatch in our sensitivity matrix since they neither have the ability to raise prices nor are they (incrementally) substitutable for other generators that attempt to raise prices. Generators that are on-line at minimum dispatch are there for a reason other than economic supply of energy for the present demand. They may be on line for system security, reserves, reactive support, etc., or to be available to supply energy (economically) at some future time. In any case, they effectively do not contribute to the sensitivity analysis and we may choose to exclude them in our calculation and analysis of the sensitivity matrix.

Finally, we conclude this discussion by pointing out the importance of a well-chosen threshold for the clustering algorithm. It must be large enough that the separation in LMPs between clusters is truly representative of some limitation in the network, and small enough to identify exploitation of market power. Of course we would like to identify all instance of market power potential – which this method will not do – and we continue related work on examination of the sensitivity matrices using spectral approaches.

VI. Acknowledgment

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Appendix: B-factor calculation

There is no unique way to calculate the B-factors that are used in the estimation of *M*. We follow a traditional approach in which the line flows and power injections are approximated using a lossless linearized system and the losses are approximated as a quadratic function of the generator injections (see, for example,[11]):

$$P_{Loss}(g) = g^T F g \qquad \text{(A1)}$$

from which the B-factors are simply

$$B = 2F \tag{A2}$$

First we start with an approximation for the total lines losses in terms of bus angles,

$$P_{Loss}(\theta) = \theta^T A^T G_{branch} A \theta \tag{A3}$$

where G_{branch} is a diagonal matrix of branch conductances and A is the branch-node incidence matrix. One can derive (A3) as the sum of branch losses assuming nominal voltages near 1pu and truncating a cosine approximation at the quadratic term. To transform (A3) into (A1) we need to express the angles in terms of the bus injections. For this we use a lossless linearized angle model. (This is somewhat inconsistent, and it can only be expected to give an approximate result in the end.) In the following expression we separate the generator injections, g, the load injections, P_d, and the reference bus injection, P_r. We also make the same distinction among angle variables. The power injections in terms of angles are

$$\begin{bmatrix} g \\ P_d \\ P_r \end{bmatrix} = \begin{bmatrix} D_{gg} & D_{gd} & D_{gr} \\ D_{dg} & D_{dd} & D_{dr} \\ D_{rg} & D_{rd} & D_{rr} \end{bmatrix} \begin{bmatrix} \theta_g \\ \theta_d \\ 0 \end{bmatrix} \tag{A4}$$

To proceed we must make some assumptions about a "slack" bus. It turns out that the resulting B-factors will depend on the choice of slack, and while in practice the matrix M appears robust to variations in B, it is important to pick a sensible slack. (See [4] for a discussion of the role of slack bus in marginal loss calculations.) Since we want a description of losses in terms of all the generators, we assign the reference bus to be at a load bus. In the spirit of a distributed slack, we enforce a condition in which any change in load must be in constant proportion to the nominal load,

$$\left(P_d^* + \Delta P_d\right) = \frac{P_d^*}{P_r^*}\left(P_r^* + \Delta P_r\right) \tag{A5}$$

(We require a choice of reference bus that has non-zero load present.) Based on the condition expressed in (A5), we left multiply (A4) by the matrix

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & -\alpha \\ 0 & 0 & 1 \end{bmatrix}$$

where $\alpha = P_d^*/P_r^*$. This yields

$$\begin{bmatrix} g \\ 0 \\ P_r \end{bmatrix} = \begin{bmatrix} D_{gg} & D_{gd} & D_{gr} \\ D_{dg} - \alpha D_{rg} & D_{dd} - \alpha D_{rd} & D_{dr} - \alpha Drr \\ D_{rg} & D_{rd} & D_{rr} \end{bmatrix} \begin{bmatrix} \theta_g \\ \theta_d \\ 0 \end{bmatrix} \tag{A6}$$

Partial inversion of the matrix gives an expression for angles in terms of generator injections:

$$\begin{bmatrix} \theta_g \\ \theta_d \\ 0 \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} D_{gg} & D_{gd} \\ D_{dg} - \alpha D_{rg} & D_{dd} - \alpha D_{dr} \end{bmatrix}^{-1} \\ 0 \quad 0 \end{bmatrix} \begin{bmatrix} g \\ 0 \end{bmatrix}$$

$$= D_{-1} g \tag{A7}$$

With (A7) we get our expression for (A1) from (A3),

$$P_{Loss}(g) = g^T D_{-1}^T A^T G_{branch} A D_{-1} g$$
$$= g^T F g$$

The expression for the B-factors is then

$$B = 2 g^T D_{-1}^T A^T G_{branch} A D_{-1} \tag{A8}$$

Using (A7) we can also get an approximation for the shift factors that describe branch flows in terms of generator injections,

$$P_{branch} = B_{branch} A \theta$$
$$= B_{branch} A D_{-1} g$$
$$= W g$$

where B_{branch} is a diagonal matrix of branch susceptances.