

Generalized Line Outage Distribution Factors

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Abstract— Distribution factors play a key role in many system security analysis and market applications. The injection shift factors (ISFs) are the basic factors that serve as building blocks of the other distribution factors. The line outage distribution factors (LODFs) may be computed using the ISFs and, in fact, may be iteratively evaluated when more than one line outage is considered. The prominent role of cascading outages in recent black-outs has created a need in security applications for evaluating LODFs under multiple-line outages. In this letter, we present an analytic, closed-form expression for and the computationally efficient evaluation of LODFs under multiple-line outages.

Index Terms—power transfer distribution factors, line outage distribution factors, multiple-line outages, system security.

I. INTRODUCTION

Distribution factors are linear approximations of the sensitivities of specific system variables with respect to changes in nodal injections and withdrawals [1]-[5]. While the line outage distribution factors (LODFs) are well understood [1], the evaluation of LODFs under multiple-line outages has received little attention. Given the usefulness of LODFs in the study of security with many outaged lines, such as in blackouts impacting large geographic regions, we focus on the fast evaluation of LODFs under multiple-line outages – the generalized LODFs or GLODFs. This letter presents an analytic, closed-form expression for, and the computationally efficient evaluation of, GLODFs.

II. BASIC DISTRIBUTION FACTORS

We consider a power system consisting of $(N+1)$ buses and L lines. We denote by $\mathcal{N}=\{0,1,\dots,N\}$ the set of buses, with the bus 0 being the slack bus, and by $\mathcal{L}=\{\ell_1,\dots,\ell_L\}$ the set of transmission lines. We associate with each line $\ell_m \in \mathcal{L}$, the ordered pair of nodes (i_m, j_m) . We use the convention that the direction of the real power flow f_{ℓ_m} on the line ℓ_m is from i_m to j_m .

The ISF $\mathbf{y}_{\ell_k}^i$ of line ℓ_k is the (approximate) sensitivity of the change in the line ℓ_k real power flow f_{ℓ_k} with respect to a change in the injection p_i at some node $i \in \mathcal{N}$ and the withdrawal of an equal change amount at the slack bus. Under the lossless conditions and the typical assumptions used in DC power flow, we construct the ISF matrix $\underline{\mathbf{Y}} \triangleq \underline{\mathbf{B}}_d \underline{\mathbf{A}} \underline{\mathbf{B}}^{-1}$ [5], with

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with $\underline{\mathbf{B}}_d \in \mathbb{R}^{L \times L}$ being the branch susceptance matrix, $\underline{\mathbf{A}} \in \mathbb{R}^{L \times N}$ the reduced incidence matrix and $\underline{\mathbf{B}} \in \mathbb{R}^{N \times N}$ the reduced nodal susceptance matrix.

We evaluate the power transfer distribution factors (PTDFs) by introducing notation for transactions. The impact of a w transaction from node i to node j , denoted by the ordered triplet $w \triangleq \{i, j, \Delta t\}$, on f_{ℓ_k} is $\Delta f_{\ell_k}^w$, and is determined by

$$\Delta f_{\ell_k}^w = \mathbf{j}_{\ell_k}^w \Delta t,$$

where the PTDF $\mathbf{j}_{\ell_k}^w$ is defined as [5]

$$\mathbf{j}_{\ell_k}^w \triangleq \mathbf{Y}_{\ell_k}^i - \mathbf{Y}_{\ell_k}^j. \quad (2)$$

For the line ℓ_m outage, we evaluate the impact $\Delta f_{\ell_k}^{(\ell_m)}$ on the flow f_{ℓ_k} on line ℓ_k using the LODF $\mathbf{V}_{\ell_k}^{(\ell_m)}$ which specifies the fraction of the pre-outage real power flow on the line ℓ_m redistributed to the line ℓ_k [5] and is given by:

$$\mathbf{V}_{\ell_k}^{(\ell_m)} \triangleq \frac{\Delta f_{\ell_k}^{(\ell_m)}}{f_{\ell_m}} = \frac{\mathbf{j}_{\ell_k}^{w(\ell_m)}}{(1 - \mathbf{j}_{\ell_m}^{w(\ell_m)})}, \ell_k \neq \ell_m. \quad (3)$$

Here, $w(\ell_m) = \{i_m, j_m, \Delta t\}$ denotes the transaction between the terminal nodes of ℓ_m . As long as $\mathbf{j}_{\ell_k}^{w(\ell_m)} \neq 1$, $\mathbf{V}_{\ell_k}^{(\ell_m)}$ is defined. The line ℓ_m outage results in a topology change and necessitates reevaluation of the post-outage network PTDFs.

We use the notation $(\mathbf{t})^{(\ell_m)}$ to denote the value of the variable \mathbf{t} with the line ℓ_m outaged, as in (3). The pre- and post-outage PTDFs, $\mathbf{j}_{\ell_k}^w$ and $(\mathbf{j}_{\ell_k}^w)^{(\ell_m)}$, respectively, are related by [5]

$$(\mathbf{j}_{\ell_k}^w)^{(\ell_m)} \triangleq \mathbf{j}_{\ell_k}^w + \mathbf{V}_{\ell_k}^{(\ell_m)} \mathbf{j}_{\ell_m}^w. \quad (4)$$

We use the distribution factors introduced in this section to generalize the LODF expression for multiple-line outages.

III. DERIVATION OF GLODFs

We first revisit the single-line outage case and examine how the outage impacts may be simulated by net injection and withdrawal changes. The line $\tilde{\ell}_1 = (\tilde{i}_1, \tilde{j}_1)$ outage changes the real power flow in the post-outage network on each line connected to \tilde{i}_1 by the fraction of $f_{\tilde{\ell}_1}$. We simulate this impact by introducing $w(\tilde{\ell}_1) = \{\tilde{i}_1, \tilde{j}_1, \Delta t(\tilde{\ell}_1)\}$ in the pre-outage network. The injection $\Delta t(\tilde{\ell}_1)$ adds a change $\mathbf{j}_{\tilde{\ell}_1}^{w(\tilde{\ell}_1)} \Delta t(\tilde{\ell}_1)$ on the line $\tilde{\ell}_1$ flow and a net flow change of $(1 - \mathbf{j}_{\tilde{\ell}_1}^{w(\tilde{\ell}_1)}) \Delta t(\tilde{\ell}_1)$ on all the other lines but $\tilde{\ell}_1$ that are connected to node \tilde{i}_1 . By selecting $\Delta t(\tilde{\ell}_1)$ to satisfy

$$(1 - \mathbf{j}_{\tilde{\ell}_1}^{w(\tilde{\ell}_1)}) \Delta t(\tilde{\ell}_1) = f_{\tilde{\ell}_1}, \quad (5)$$

the transaction $w(\tilde{\ell}_1)$ changes the flow f_{ℓ_k} , $\ell_k \neq \tilde{\ell}_1$, by

$$\Delta f_{\ell_k}^{w(\tilde{\ell}_1)} = \mathbf{j}_{\ell_k}^{w(\tilde{\ell}_1)} \Delta t(\tilde{\ell}_1) = \left[\mathbf{j}_{\ell_k}^{w(\tilde{\ell}_1)} (1 - \mathbf{j}_{\tilde{\ell}_1}^{w(\tilde{\ell}_1)})^{-1} \right] f_{\tilde{\ell}_1}. \quad (6)$$

In terms of (3), the bracketed term in (6) is $\mathbf{V}_k^{(\tilde{\ell}_1)}$, and so $w(\tilde{\ell}_1)$ with $\Delta t(\tilde{\ell}_1)$ given by (6) simulates the line $\tilde{\ell}_1$ outage impacts.

We proceed with the generalization for multiple-line outages by next considering the case of the outages of the two lines $\tilde{\ell}_1$ and $\tilde{\ell}_2$. We simulate the impacts on f_{ℓ_k} , by introducing $w(\tilde{\ell}_1)$ and $w(\tilde{\ell}_2)$, taking explicitly into account the interactions between these two transactions in specifying $\Delta t(\tilde{\ell}_1)$ and $\Delta t(\tilde{\ell}_2)$, as shown in Fig. 1. We set $\Delta t(\tilde{\ell}_1)$ to satisfy

$$\left(1 - \left(\mathbf{j}_{\tilde{\ell}_1}^{w(\tilde{\ell}_1)}\right)^{(\tilde{\ell}_2)}\right) \Delta t(\tilde{\ell}_1) = \left(f_{\tilde{\ell}_1}\right)^{(\tilde{\ell}_2)}. \quad (7)$$

Analogously, we select $\Delta t(\tilde{\ell}_2)$ to satisfy

$$\left(1 - \left(\mathbf{j}_{\tilde{\ell}_2}^{w(\tilde{\ell}_2)}\right)^{(\tilde{\ell}_1)}\right) \Delta t(\tilde{\ell}_2) = \left(f_{\tilde{\ell}_2}\right)^{(\tilde{\ell}_1)}. \quad (8)$$

We rewrite (7) and (8) using the relations in (3) and (4) as

$$\left[\mathbf{I} - \begin{bmatrix} \mathbf{j}_{\tilde{\ell}_1}^{w(\tilde{\ell}_1)} & \mathbf{j}_{\tilde{\ell}_1}^{w(\tilde{\ell}_2)} \\ \mathbf{j}_{\tilde{\ell}_2}^{w(\tilde{\ell}_1)} & \mathbf{j}_{\tilde{\ell}_2}^{w(\tilde{\ell}_2)} \end{bmatrix} \right] \begin{bmatrix} \Delta t(\tilde{\ell}_1) \\ \Delta t(\tilde{\ell}_2) \end{bmatrix} = \begin{bmatrix} f_{\tilde{\ell}_1} \\ f_{\tilde{\ell}_2} \end{bmatrix}. \quad (9)$$

As long as the matrix in (9) is nonsingular, we determine $\Delta t(\tilde{\ell}_1)$ and $\Delta t(\tilde{\ell}_2)$ by solving the linear system.

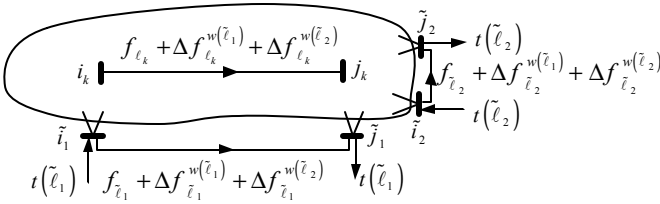


Fig. 1. The impacts of the transactions $w(\tilde{\ell}_1)$ and $w(\tilde{\ell}_2)$.

In the inductive process to generalize the result for the case of multiple-line outages, we assume that the impacts of a set of $a-1$ outaged lines $\tilde{\mathcal{L}}_{(a-1)} = \{\tilde{\ell}_1, \dots, \tilde{\ell}_{a-1}\}$ are simulated with $a-1$ transactions whose amounts are specified by

$$\left[\mathbf{I} - \mathbf{F}_{\tilde{\mathcal{L}}_{(a-1)}} \right] \underline{\Delta \mathbf{t}}_{(a-1)} = \underline{\mathbf{f}}_{(a-1)}, \quad (10)$$

where, $\underline{\Delta \mathbf{t}}_{(a-1)} = [\Delta t(\tilde{\ell}_1), \dots, \Delta t(\tilde{\ell}_{a-1})]^T$, $\underline{\mathbf{f}}_{(a-1)} = [f_{\tilde{\ell}_1}, \dots, f_{\tilde{\ell}_{a-1}}]^T$

and $\mathbf{F}_{\tilde{\mathcal{L}}_{(a-1)}} = \begin{bmatrix} \mathbf{j}_{\tilde{\ell}_1}^{w(\tilde{\ell}_1)} & \dots & \mathbf{j}_{\tilde{\ell}_1}^{w(\tilde{\ell}_{a-1})} \\ \vdots & \ddots & \vdots \\ \mathbf{j}_{\tilde{\ell}_{a-1}}^{w(\tilde{\ell}_1)} & \dots & \mathbf{j}_{\tilde{\ell}_{a-1}}^{w(\tilde{\ell}_{a-1})} \end{bmatrix}$, with $[\mathbf{I} - \mathbf{F}_{\tilde{\mathcal{L}}_{(a-1)}}]$ nonsingular.

We now consider the additional line $\tilde{\ell}_a \notin \tilde{\mathcal{L}}_{(a-1)}$ outage. The set of outaged lines is $\tilde{\mathcal{L}}_{(a)} = \tilde{\mathcal{L}}_{(a-1)} \cup \{\tilde{\ell}_a\}$. Reasoning along the lines used in the two-line outage analysis, $\underline{\Delta \mathbf{t}}_{(a-1)}$ is given by

$$\left[\mathbf{I} - \left(\mathbf{F}_{\tilde{\mathcal{L}}_{(a-1)}} \right)^{(\tilde{\ell}_a)} \right] \underline{\Delta \mathbf{t}}_{(a-1)} = \left(\underline{\mathbf{f}}_{(a-1)} \right)^{(\tilde{\ell}_a)}. \quad (11)$$

We capture the impacts of the outages of the $\tilde{\mathcal{L}}_{(a-1)}$ elements on $\tilde{\ell}_a$ by using the analogue of (8) and determine $\Delta t(\tilde{\ell}_a)$ from

$$\left(1 - \left(\mathbf{j}_{\tilde{\ell}_a}^{w(\tilde{\ell}_a)}\right)^{(\tilde{\mathcal{L}}_{(a-1)})}\right) \Delta t(\tilde{\ell}_a) = \left(f_{\tilde{\ell}_a}\right)^{(\tilde{\mathcal{L}}_{(a-1)})}. \quad (12)$$

The superscript $(\tilde{\mathcal{L}}_{(a-1)})$ denotes the network with the

elements of $\tilde{\mathcal{L}}_{(a-1)}$ outaged. We define $\underline{\mathbf{b}} \triangleq [\mathbf{j}_{\tilde{\ell}_1}^{w(\tilde{\ell}_a)}, \dots, \mathbf{j}_{\tilde{\ell}_{a-1}}^{w(\tilde{\ell}_a)}]^T$ and $\underline{\mathbf{c}} \triangleq [\mathbf{j}_{\tilde{\ell}_a}^{w(\tilde{\ell}_1)}, \dots, \mathbf{j}_{\tilde{\ell}_a}^{w(\tilde{\ell}_{a-1})}]^T$ and rewrite (11) and (12) as

$$\begin{aligned} \left[\mathbf{I} - \mathbf{F}_{\tilde{\mathcal{L}}_{(a-1)}} \right] \underline{\Delta \mathbf{t}}_{(a-1)} - \underline{\mathbf{b}} \left(1 - \mathbf{j}_{\tilde{\ell}_a}^{w(\tilde{\ell}_a)}\right)^{-1} \left(f_{\tilde{\ell}_a} + \underline{\mathbf{c}}^T \underline{\Delta \mathbf{t}}_{(a-1)}\right) &= \underline{\mathbf{f}}_{(a-1)} \\ \left(1 - \mathbf{j}_{\tilde{\ell}_a}^{w(\tilde{\ell}_a)}\right) \Delta t(\tilde{\ell}_a) - \underline{\mathbf{c}}^T \left[\mathbf{I} - \mathbf{F}_{\tilde{\mathcal{L}}_{(a-1)}} \right]^{-1} \left(\underline{\mathbf{f}}_{(a-1)} + \underline{\mathbf{b}} \Delta t(\tilde{\ell}_a)\right) &= f_{\tilde{\ell}_a}, \end{aligned} \quad (13)$$

which may be simplified to

$$\left[\mathbf{I} - \mathbf{F}_{\tilde{\mathcal{L}}_{(a)}} \right] \underline{\Delta \mathbf{t}}_{(a)} = \underline{\mathbf{f}}_{(a)}, \quad \mathbf{F}_{\tilde{\mathcal{L}}_{(a)}} \triangleq \begin{bmatrix} \mathbf{F}_{\tilde{\mathcal{L}}_{(a-1)}} & \underline{\mathbf{b}} \\ \underline{\mathbf{c}}^T & \mathbf{j}_{\tilde{\ell}_a}^{w(\tilde{\ell}_a)} \end{bmatrix}. \quad (14)$$

So long as $[\mathbf{I} - \mathbf{F}_{\tilde{\mathcal{L}}_{(a)}}]$ is nonsingular, we use (14) to solve for $\underline{\Delta \mathbf{t}}_{(a)}$ and so simulate the impacts of the \mathbf{a} line outages.

This development for specifying the appropriate values of the transactions is used to provide the *GLODF* expression. For any line $\ell_k \notin \tilde{\mathcal{L}}_{(a)}$, we define $\underline{\mathbf{x}}_{\ell_k}^{(\tilde{\mathcal{L}}_{(a)})}$, whose elements are the *GLODFs* with the lines in $\tilde{\mathcal{L}}_{(a)}$ outaged, with the interactions between the outaged lines fully considered. The change in the real power flow of line ℓ_k is

$$\left(\Delta f_{\ell_k}\right)^{(\tilde{\mathcal{L}}_{(a)})} \triangleq \left[\underline{\mathbf{x}}_{\ell_k}^{(\tilde{\mathcal{L}}_{(a)})}\right]^T \underline{\mathbf{f}}_{(a)}, \quad \ell_k \notin \tilde{\mathcal{L}}_{(a)} \quad (15)$$

However, the combined impacts on line ℓ_k of the \mathbf{a} transactions with the $\underline{\Delta \mathbf{t}}_{(a)}$ specified by (14) is

$$\left(\Delta f_{\ell_k}\right)^{(\tilde{\mathcal{L}}_{(a)})} = \left[\mathbf{j}_{\ell_k}^{w(\tilde{\ell}_1)}, \dots, \mathbf{j}_{\ell_k}^{w(\tilde{\ell}_a)}\right] \underline{\Delta \mathbf{t}}_{(a)}. \quad (16)$$

Therefore, for $[\mathbf{I} - \mathbf{F}_{\tilde{\mathcal{L}}_{(a)}}]$ nonsingular, we rewrite (16) as

$$\left(\Delta f_{\ell_k}\right)^{(\tilde{\mathcal{L}}_{(a)})} = \left[\mathbf{j}_{\ell_k}^{w(\tilde{\ell}_1)}, \dots, \mathbf{j}_{\ell_k}^{w(\tilde{\ell}_a)}\right] \left[\mathbf{I} - \mathbf{F}_{\tilde{\mathcal{L}}_{(a)}}\right]^{-1} \underline{\mathbf{f}}_{(a)}. \quad (17)$$

It follows from (15) that $\underline{\mathbf{x}}_{\ell_k}^{(\tilde{\mathcal{L}}_{(a)})}$ is the solution of

$$\left[\mathbf{I} - \mathbf{F}_{\tilde{\mathcal{L}}_{(a)}}\right]^T \underline{\mathbf{x}}_{\ell_k}^{(\tilde{\mathcal{L}}_{(a)})} = \left[\mathbf{j}_{\ell_k}^{w(\tilde{\ell}_1)}, \dots, \mathbf{j}_{\ell_k}^{w(\tilde{\ell}_a)}\right]^T, \quad (18)$$

and is defined whenever $[\mathbf{I} - \mathbf{F}_{\tilde{\mathcal{L}}_{(a)}}]$ is nonsingular. The case for singular $[\mathbf{I} - \mathbf{F}_{\tilde{\mathcal{L}}_{(a)}}]$ indicates that the outage of the \mathbf{a} lines in $\tilde{\mathcal{L}}_{(a)}$ separates the system into two or more islands. The analysis of such cases is treated in [6]. In fact, a simple case is the outage of a line whose *PTDF* equals one. In this case, the outage of the line results in the creation of two separate subsystems. When the outaged line whose *PTDF* is unity happens to be a radial tie, the outage results in the isolation of the radial node.

The relation (18) provides an analytic, closed-form expression for the *GLODFs*. Since the *GLODF* is expressed in terms of the pre-outage network parameters, we avoid the need to evaluate the post-outage network parameters. A key advantage in the deployment of *GLODFs* is the ability to evaluate the post-outage flows on specific lines of interest without the need to determine the post-outage network states. The proposed *LODF* extension permits the *GLODF* evaluation through a computationally efficient procedure which involves the solution of a system of linear equations whose dimension is

is the number of line outages.

IV. SUMMARY

The prominent role of cascading outages in recent blackouts has created a critical need in security applications for the rapid assessment of multiple-line outage impacts. We developed a closed-form analytic expression for *GLODFs* under multiple-line outages without the reevaluation of post-outage network system parameters. This general expression allows the computationally efficient evaluation of *GLODFs* for security application purposes. A very useful application of *GLODFs* is in the detection of island formation and the identification of causal factors under multiple-line outages [6].

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