

The dynamics of power system markets

Fernando L. Alvarado

Department of Electrical and Computer Engineering, 1415 Engineering Drive
The University of Wisconsin, Madison, Wisconsin 53706
Phone: 608-262-8900 FAX: 608-265-6651
e-mail: alvarado@engr.wisc.edu

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Abstract

Market equilibrium conditions can be derived from more general dynamic equations describing the marketplace. Dynamic market equations provide additional insights into the behavior and stability of markets which are not available from static models. For example, markets with a single supplier with declining linear costs (economies of scale) may or may not be stable, depending on specific cost characteristics. Markets with more than one supplier with declining linear costs are always unstable. Energy imbalance requires careful design: a combination of aggressive energy imbalance regulation and rapid response of the market can lead to unstable market conditions. Congestion has the potential of substantially altering the market equilibrium point. However, if properly handled, it does not affect market stability.

1 Introduction

When designing Power Exchanges and policies for Independent System Operators for Electric Power Systems, it is necessary to consider whether the markets that underlie this mode of operation lead to well-behaved (stable) market equilibrium conditions. This report studies the impact of various policies on the dynamic behavior of power system markets. The impact of dynamics has been used with success by economists to better understand markets¹.

Several cases of increasing complexity are considered: the classic steady state results for economic equilibria are readily derived for each case, along with conditions for market stability. The cases considered are:

- A one-supplier one-consumer case.
- A fixed (inelastic) demand case, first with two and then with m suppliers.
- A case with m suppliers and n elastic (price-responsive) consumers.
- A case with m suppliers, n consumers, and commodity (energy) storage capability.
- The effect of network congestion on a market with several suppliers and several consumers.

A set of simple numerical examples is presented and solved to illustrate the points at hand. A number of principles are derived, and the implications for the operation of electric power markets are explained.

Certain assumptions are made throughout the report. The purpose of these assumptions is to facilitate the introduction of concepts. The assumptions are:

- Marginal production costs λ_g are linear functions of the output quantity² (the power P_g).
- Marginal benefit functions λ_d are negatively-sloping linear functions of power consumption P_d .
- Any suppliers/consumers at a limit are considered as fixed injections/extractions to/from the market.
- Response of both suppliers and consumers to observed prices is not instantaneous. They are governed by first order single time constant differential equations.
- Except as otherwise noted, supply and demand are in precise balance at all times (there is no storage of energy).
- Except as otherwise noted, there are no network constraints.
- Transportation (transmission) losses are negligible.

¹See, for example, Leontief [1, 2] and Hawkins [3].

²With due apologies to economists who may read this report, the symbol P is used to denote quantity (**power**), not **price**. Price is denoted by λ .

2 The dynamics of simple economic systems

If a supplier observes a market price λ above his/her production cost λ_{gi} , it is assumed that the supplier will expand production until the marginal cost of production equals the price. The rate of expansion is proportional to the difference between the observed price and the actual production cost. The speed with which the generation power output P_{gi} of supplier i can respond is supplier dependent. It is denoted by a time constant τ_{gi} for supplier i . Let the price at any given time be λ . The above yields the following differential equation:

$$\tau_{gi}\dot{P}_{gi} = \lambda - b_{gi} - c_{gi}P_{gi} \quad (1)$$

where $b_{gi} + c_{gi}P_{gi}$ is the marginal cost λ_{gi} of supplier i .

A consumer demand P_{di} with a marginal benefit function λ_{di} above the marginal price will expand consumption until parity is attained. The speed of expansion is consumer dependent, and it is characterized by a time constant τ_{di} . The equation describing the behavior of a consumer is (the subscript i is implicit):

$$\tau_{di}\dot{P}_{di} = b_{di} + c_{di}P_{di} - \lambda \quad (2)$$

where $b_{di} + c_{di}P_{di}$ is the marginal consumer benefit.

The final condition required to characterize the marketplace is balance between supply and demand. If there is no energy storage, such a condition for the case of m suppliers and n consumers is characterized by:

$$\sum_{i=1}^m P_{gi} = \sum_{i=1}^n P_{di} \quad (3)$$

which specializes trivially to the case of one supplier and one consumer:

$$P_g = P_d \quad (4)$$

If a certain amount of energy can be stored in the system (as long as under steady-state conditions no such energy remains because it is driven to zero), the power balance equation 4 must be modified to include the dynamics of the stored energy. The energy is determined from:

$$\dot{E} = P_g - P_d \quad (5)$$

Stored energy is the integral of power imbalance. Or, equivalently, the derivative of the stored energy (E) is equal to the power imbalance. Suitable equations are later added to drive the steady-state stored energy E to zero.

The models in this report are by nature stochastic. For many studies, the stochastic nature of the problem can be represented by considering b_g and b_d to be random variables. The present analysis is limited to the case of deterministic

variables and costs, or rather, to the case where the expected values of stochastic variables are used.

The models in this paper can also be extended quite naturally to take into consideration the transition from long-term to short-term decisions. In particular, long run and short run responses will differ for both consumers and producers. Both c_g and c_d will generally have a smaller absolute value in the long run case. The transition from long run to short run costs can be achieved by making the marginal benefit (and/or the marginal cost) adjustable according to their own dynamic equations, which depend on the dynamics of the cost of capital and the benefits of system expansion.

3 A one-supplier one-consumer case

Consider the set of differential/algebraic equations (DAE) corresponding to the case of one supplier with demand elasticity c_g and one consumer with demand elasticity c_d . The DAEs characterizing this case are:

$$\tau_g \dot{P}_g = \lambda - b_g - c_g P_g \quad (6)$$

$$\tau_d \dot{P}_d = b_d + c_d P_d - \lambda \quad (7)$$

$$P_g = P_d \quad (8)$$

which, in matrix form, can be expressed as:

$$\begin{bmatrix} \tau_g & 0 & 0 \\ 0 & \tau_d & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{P}_g \\ \dot{P}_d \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -c_g & 0 & 1 \\ 0 & +c_d & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} P_g \\ P_d \\ \lambda \end{bmatrix} + \begin{bmatrix} -b_g \\ b_d \\ 0 \end{bmatrix} \quad (9)$$

The interpretation of these equations is as follows: the supplier acts in a way that tends to increase production when prices exceed production marginal costs. The consumer acts in a way that tends to increase consumption when marginal benefits exceed price.

For every case in this report, the static equilibrium point is determined first, followed by an analysis of the dynamic stability of such an equilibrium point.

The equilibrium for the DAE system above is obtained by setting the derivative terms to zero:

$$0 = \lambda - b_g - c_g P_g \quad (10)$$

$$0 = b_d + c_d P_d - \lambda \quad (11)$$

$$0 = P_g - P_d \quad (12)$$

which, expressed in matrix form, leads to:

$$\begin{bmatrix} -c_g & 0 & 1 \\ 0 & c_d & -1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} P_g \\ P_d \\ \lambda \end{bmatrix} = \begin{bmatrix} b_g \\ -b_d \\ 0 \end{bmatrix} \quad (13)$$

Solution of this algebraic linear problem leads to:

$$P_g = \frac{b_d - b_g}{c_g - c_d} \quad (14)$$

$$P_d = P_g \quad (15)$$

$$\lambda = \frac{-b_g c_d + c_g b_d}{c_g - c_d} \quad (16)$$

The following assumptions are reasonable:

- Assume that $c_d < 0$, indicating that marginal consumer benefit decreases with consumption.
- Assume that $b_d > b_g$, indicating that initial consumer marginal benefit is greater than initial producer marginal cost.
- Assume that $b_g > 0$ (“no free lunch”).
- While it can generally be assumed that $c_g > 0$, the case of $c_g \leq 0$ (“economies of scale” for the supplier) cannot always be ruled out. Both cases are thus considered here.

Under these assumptions, the following observations apply to this solution:

- The solution occurs when marginal cost of production equals marginal benefit of consumption.
- The equilibrium point can be shown to result in a positive price $\lambda > 0$ as well as a positive amount of power delivered $P_g = P_d > 0$.

For an analysis of the dynamics of this system, the price λ can be eliminated from the original dynamic equations by adding equations 6 and 7. This DAE model represents a first order system³. Substitution from 8, re-arrangement, and retention of the homogeneous portion of the result leads to:

$$(\tau_g + \tau_d)\dot{P}_g = -(c_g - c_d)P_g \quad (17)$$

The condition for the stability of this equation is that the eigenvalue associated with this problem be negative, that is:

$$\frac{c_g - c_d}{\tau_g + \tau_d} > 0 \quad (18)$$

or simply, since $\tau_g > 0$ and $\tau_d > 0$:

$$c_g > c_d \quad (19)$$

³Implicit in this DAE model is the assumption that, although response of a market to a price signal is not instantaneous, such a response must result in power balance at all times.

Satisfying this condition is virtually assured, since $c_d < 0$ (quite often $c_d \ll 0$). Thus, this type of market is stable, and static analysis is sufficient to ascertain the economic behavior of such a market.

The reason that so much effort has been taken to reach such an intuitive conclusion is because the interest in developing a general method suitable for any size system and any set of similar assumptions. More complex cases can now be studied using the same eigenvalue method.

4 A fixed-demand case

The DAEs characterizing the two-supplier fixed-demand case are:

$$\tau_{g1} \dot{P}_{g1} = \lambda - b_{g1} - c_{g1} P_{g1} \quad (20)$$

$$\tau_{g2} \dot{P}_{g2} = \lambda - b_{g2} - c_{g2} P_{g2} \quad (21)$$

$$P_{g1} + P_{g2} = P_D \quad (22)$$

which, in matrix form, can be expressed as:

$$\begin{bmatrix} \tau_{g1} & 0 & 0 \\ 0 & \tau_{g2} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{P}_{g1} \\ \dot{P}_{g2} \\ \lambda \end{bmatrix} = \begin{bmatrix} -c_{g1} & 0 & 1 \\ 0 & -c_{g2} & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ \lambda \end{bmatrix} + \begin{bmatrix} -b_{g1} \\ -b_{g2} \\ -P_D \end{bmatrix} \quad (23)$$

The equilibrium point is obtained from the solution of the following linear set of equations:

$$\begin{bmatrix} -c_{g1} & 0 & 1 \\ 0 & -c_{g2} & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ \lambda \end{bmatrix} = \begin{bmatrix} b_{g1} \\ b_{g2} \\ -P_D \end{bmatrix} \quad (24)$$

This solution generalizes readily to the case of m suppliers and fixed demand:

$$\begin{bmatrix} -c_{g1} & 0 & \cdots & 0 & 1 \\ 0 & -c_{g2} & \cdots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & -c_{gm} & 1 \\ 1 & 1 & \cdots & 1 & 0 \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ \vdots \\ P_{gm} \\ \lambda \end{bmatrix} = \begin{bmatrix} b_{g1} \\ b_{g2} \\ \vdots \\ b_{gm} \\ P_D \end{bmatrix} \quad (25)$$

The following are observations from the steady state equilibrium conditions:

- Equilibrium occurs when all suppliers which are not constrained by either minimum or maximum limits operate at the same marginal cost (λ).
- There are no a-priori assurances that the solution of equation 25 results in all suppliers with positive output at the equilibrium point⁴.

⁴Assume that some producers would have negative output according to equations 25. This means that these producers will not participate in the market.

Dynamic analysis of this problem can be done by eliminating (subtracting) the first dynamic equation (equation 20) from the second (equation 21). After this is done, the variable P_{g1} is eliminated by substitution from equation 22. The homogeneous part of the resulting differential equation is:

$$(\tau_{g1} + \tau_{g2})\dot{P}_{g2} = -(c_{g1} + c_{g2})P_{g2} \quad (26)$$

Observations from this equation are:

- Provided a solution exists, its stability is independent of the linear cost coefficients b_{g1} and b_{g2} .
- If both $c_{g1} > 0$ and $c_{g2} > 0$, stability is assured.
- It is sufficient to have $c_{g1} + c_{g2} > 0$ to assure stability. Thus, it is possible for either c_1 or c_2 to be zero or slightly negative (but not both) and still have stable market conditions.

Generalization to the case of m suppliers follows the same steps. The resulting homogeneous equations (in matrix form) are:

$$\begin{bmatrix} \tau_{g1} + \tau_{g2} & \tau_{g1} & \cdots & \tau_{g1} \\ \tau_{g1} & \tau_{g1} + \tau_{g3} & \cdots & \tau_{g1} \\ \vdots & \vdots & \ddots & \vdots \\ \tau_{g1} & \tau_{g1} & \cdots & \tau_{g1} + \tau_{gm} \end{bmatrix} \begin{bmatrix} \dot{P}_{g2} \\ \dot{P}_{g3} \\ \vdots \\ \dot{P}_{gm} \end{bmatrix} = \begin{bmatrix} c_{g1} + c_{g2} & c_{g1} & \cdots & c_{g1} \\ c_{g1} & c_{g1} + c_{g3} & \cdots & c_{g1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{g1} & c_{g1} & \cdots & c_{g1} + c_{gm} \end{bmatrix} \begin{bmatrix} P_{g2} \\ P_{g3} \\ \vdots \\ P_{gm} \end{bmatrix} \quad (27)$$

The dynamic characteristics of the market are dictated by the eigenvalues of this generalized eigenvalue problem. The following additional general observations can be made from the diagonal-dominance characteristics of these matrices:

- If $c_{gi} > 0$ for all i , the result is a stable market (all eigenvalues are negative).
- If $c_{gi} \leq 0$ for two or more values of i , the result is necessarily an unstable market.
- It is sufficient to ascertain the stability properties of the right hand side matrix to determine the stability properties of the solution.

The practical consequence of the above is that markets can tolerate *one* supplier that exhibits economies of scale ($c_i < 0$), provided the economies are

$$\begin{bmatrix} -c_{g1} & & & & & & & 1 \\ & \ddots & & & & & & \vdots \\ & & -c_{gm} & & & & & 1 \\ & & & +c_{d1} & & & & -1 \\ & & & & \ddots & & & \vdots \\ & & & & & +c_{dn} & & -1 \\ 1 & \cdots & 1 & -1 & \cdots & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{g1} \\ \vdots \\ P_{gm} \\ P_{d1} \\ \vdots \\ P_{dn} \\ \lambda \end{bmatrix} + \begin{bmatrix} -b_{g1} \\ \vdots \\ -b_{gm} \\ +b_{d1} \\ \vdots \\ +b_{dn} \\ 0 \end{bmatrix} \quad (28)$$

The static equilibrium for this problem can be obtained from the solution of the following algebraic problem:

$$\begin{bmatrix} c_{g1} & & & & & & & -1 \\ & \ddots & & & & & & \vdots \\ & & c_{gm} & & & & & -1 \\ & & & -c_{d1} & & & & 1 \\ & & & & \ddots & & & \vdots \\ & & & & & -c_{dn} & & 1 \\ -1 & \cdots & -1 & 1 & \cdots & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} P_{g1} \\ \vdots \\ P_{gm} \\ P_{d1} \\ \vdots \\ P_{dn} \\ \lambda \end{bmatrix} = \begin{bmatrix} -b_{g1} \\ \vdots \\ -b_{gm} \\ b_{d1} \\ \vdots \\ b_{dn} \\ 0 \end{bmatrix} \quad (29)$$

And the conditions for dynamic stability of the above problem can be obtained from an inspection of the eigenvalues of the following reduced generalized eigenvalue problem:

$$\begin{bmatrix} \tau_{g1} + \tau_{g2} & \cdots & \tau_{g1} & -\tau_{g1} & \cdots & -\tau_{g1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \tau_{g1} & \cdots & \tau_{g1} + \tau_{gm} & -\tau_{g1} & \cdots & -\tau_{g1} \\ -\tau_{g1} & \cdots & -\tau_{g1} & \tau_{g1} + \tau_{d1} & \cdots & \tau_{g1} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ -\tau_{g1} & \cdots & -\tau_{g1} & \tau_{g1} & \cdots & \tau_{g1} + \tau_{dn} \end{bmatrix} \begin{bmatrix} \dot{P}_{g2} \\ \vdots \\ P_{gm} \\ P_{d1} \\ \vdots \\ \dot{P}_{dn} \end{bmatrix} = \begin{bmatrix} -c_{g1} - c_{g2} & \cdots & -c_{g1} & c_{g1} & \cdots & c_{g1} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ -c_{g1} & \cdots & -c_{g1} - c_{gm} & c_{g1} & \cdots & c_{g1} \\ c_{g1} & \cdots & c_{g1} & -c_{g1} + c_{d1} & \cdots & -c_{g1} \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ c_{g1} & \cdots & c_{g1} & -c_{g1} & \cdots & -c_{g1} + c_{dn} \end{bmatrix} \begin{bmatrix} P_{g2} \\ \vdots \\ P_{gm} \\ P_{d1} \\ \vdots \\ P_{dn} \end{bmatrix} \quad (30)$$

which, for a two-supplier, one-load system, reduces to:

$$\begin{bmatrix} \tau_{g1} + \tau_{g2} & -\tau_{g1} \\ -\tau_{g1} & \tau_{g1} + \tau_{d1} \end{bmatrix} \begin{bmatrix} \dot{P}_{g2} \\ P_d \end{bmatrix} = \begin{bmatrix} -c_{g1} - c_{g2} & c_{g1} \\ c_{g1} & -c_{g1} + c_d \end{bmatrix} \begin{bmatrix} P_{g2} \\ P_d \end{bmatrix} \quad (31)$$

The eigenvalues of this problem dictate the stability of the market. If $c_{gi} > 0$ for $1 \leq i \leq m$ and $c_{di} > 0$ for $1 \leq i \leq n$, stability is assured. This verifies

that, even if demands are elastic, the market equilibrium is stable if no supplier exhibit economies of scale and no consumer exhibits “economies of consumption”, that is, regions in their marginal benefit function where benefits increase as consumption increases.

6 The case of energy imbalance

Consider now the situation that occurs when the condition for strict power balance at all times is relaxed. Instead of requiring:

$$P_{g1} + P_{g2} + \dots + P_{gm} - P_{d1} - P_{d2} - \dots - P_{dn} = 0 \quad (32)$$

the resulting energy imbalance E is computed from:

$$\dot{E} = P_{g1} + P_{g2} + \dots + P_{gm} - P_{d1} - P_{d2} - \dots - P_{dn} \quad (33)$$

Of course, energy imbalance cannot be sustained indefinitely. It must be reduced or driven to zero. In a traditional utility environment, such an objective is attained by automatic generation control⁷. In a market-driven model, it must be assumed that prices will reflect the degree of energy imbalance. That is, an excess of the quantity supplied (the power P_g) in the grid will slightly depress the value of the power to the consumers, and thus will decrease the price they are willing to pay. Such a situation can be represented by adjusting prices depending on the degree of energy imbalance. As explained below, it may also be necessary to alter the prices seen by the suppliers whenever there is any persisting energy imbalance. This point was the basis for the proposals by Bohn et al. [5] and has been recently promoted further by [6, 7].

In what follows, we refer to the changing of price depending on excess or shortfall or real time energy balance as “frequency regulation pricing” or “ACE (Area Control Error) pricing.” The equations representing the dynamics of ACE pricing for a one-supplier one-consumer case are:

$$\tau_g \dot{P}_g = -b_g - c_g P_g + \lambda - kE \quad (34)$$

$$\tau_d \dot{P}_d = b_d + c_d P_g - \lambda \quad (35)$$

$$\dot{E} = P_g - P_d \quad (36)$$

$$\tau_\lambda \dot{\lambda} = -E \quad (37)$$

The interpretation of these equations is as follows: generators act in a way that tends to increase production when prices exceed production marginal costs. Consumers act in a way that tends to increase consumption when marginal benefits exceed price. Since it may be impossible to perfectly balance supply and consumption at all times, any discrepancy accumulates as an energy error. In a practical system, this results in either an increase in frequency or an increase in the Area Control Error (ACE). The result of an excess of energy is a reduction

⁷See [4] for an excellent reference on conventional AGC.

in the system price, which takes place according to some time constant τ_λ . This reduction in system price increases consumption and decreases production, thereby leading to a decrease in the excess energy.

Stability requirements also necessitate the presence of a supplementary stabilizing price signal to be sent to either the suppliers or the consumers. In the case considered here, the signal is sent only to the suppliers. The stabilizing signal sent is equal to a constant k times the accumulated energy error (equation 34). This signal can be interpreted as a “bias” that is added to prices whenever the energy error is nonzero. This supplementary market signal is *essential* to stable market behavior.

The equilibrium of the above dynamic market equations occurs when:

$$\begin{bmatrix} -c_g & 0 & -k & 1 \\ 0 & +c_d & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} P_g \\ P_d \\ E \\ \lambda \end{bmatrix} = \begin{bmatrix} b_g \\ -b_d \\ 0 \\ 0 \end{bmatrix} \quad (38)$$

The equilibrium for this case occurs when $P_g = P_d$, $E = 0$, and at equal marginal production cost and marginal consumer benefit.

The dynamic properties of this equilibrium point are dictated by the eigenvalues of the following generalized eigenvalue problem:

$$\begin{bmatrix} \tau_g & 0 & 0 & 0 \\ 0 & \tau_d & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \tau_\lambda \end{bmatrix} \begin{bmatrix} \dot{P}_g \\ \dot{P}_d \\ \dot{E} \\ \dot{\lambda} \end{bmatrix} = \begin{bmatrix} -c_g & 0 & -k & 1 \\ 0 & c_d & 0 & -1 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} P_g \\ P_d \\ E \\ \lambda \end{bmatrix} \quad (39)$$

Observe that this problem contains no algebraic conditions. The stability properties of this problem are controlled by the eigenvalues of equation 39.

Although not illustrated here, these equations generalize readily to the case of multiple suppliers and multiple consumers. The following observations can be made about the problem with energy imbalance:

- Energy imbalance, as long as it is eventually driven to zero, does not affect the ultimate equilibrium point of the market.
- Failure to provide price or other feedback mechanisms to the market will result in permanent energy imbalance that will make markets unable to balance supply and demand. Thus, the need for energy imbalance monitoring and correction is essential.
- Providing energy imbalance feedback based on energy imbalance alone is not enough. It will result in price instabilities. The integral of the energy imbalance (i.e., the net accumulated energy imbalance) needs to be used as a signal as well.
- Feedback gains (k in the example above) and time constants intended to drive energy imbalance toward zero can, under certain conditions, make

the market behavior unstable. The problem is most severe when large gains and small time constants are used. This will be illustrated by example.

- However, if the time constants are made too slow the system will spend a considerable amount of time with nonzero imbalance. If the costs of operating the system are not stationary with time and the system is not at a zero-imbalance condition, this fact can give rise to opportunities for market participants that are savvy of the dynamics of the market.
- It is impossible for an eigenvalue to become zero unless there are economies of scale, that is, unless some values of c_g are less than zero. Thus, the instabilities that occur are oscillatory instabilities.

All these observations are based on the nature and structure of the equations above, and will be further corroborated in the numerical examples section.

7 The effect of congestion

Congestion has the effect of imposing additional requirements on the market. In most markets, these effects are quite trivial and easy to visualize: the amounts of “goods” that can be shipped from A to B is limited, thus, the markets separate. In electric networks congestion is no less real, but it is harder to quantify and visualize because of the nature of network flows.

In the end, congestion restrictions can be characterized simply as one or more equality conditions that must be satisfied by the combination of all suppliers and consumers. Only “binding” congestion needs to be formally represented. Potential congestion needs to be monitored, but it need not be explicitly represented⁸.

The characterization of congestion is thus done by means of a sensitivity matrix structure. That is, it is possible to obtain linear expressions that relate individual power injections to individual flows. This linear equation is valuable because it characterizes all viable trading thresholds which prevent a congested line from being further congested. This notion is the basis for the current coordinated trades proposal [8].

A single congested condition can be represented as a scalar additional equality constraint:

$$S_{g1}P_{g1} + S_{g2}P_{g2} \cdots + S_{gm}P_{gm} + S_{d1}P_{d1} + S_{d2}P_{d2} \cdots + S_{dn}P_{dn} = s_1 \quad (40)$$

⁸The exception to this is when probabilistic analysis is used. In this case, congestion can always occur, and what needs to be characterized is the probability with which constraint conditions will occur. Even in this case, however, only congestion conditions that have a nontrivial probability of occurrence need to be represented.

The problem is now too complex continue using detailed matrix notation. Switching to a more compact notation:

$$\begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{P}} \\ \dot{\Lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{C} & \mathbf{S}^t \\ \mathbf{S} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{P} \\ \Lambda \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \mathbf{s} \end{bmatrix} \quad (43)$$

where:

\mathbf{T} is a diagonal matrix with the time constants for all powers, generation as well as demand.

\mathbf{P} is a vector of all powers (m generation powers and n demand powers).

Λ is a vector with all lagrange multipliers (the price λ is the first of these).

\mathbf{C} is the diagonal matrix of all quadratic cost coefficients c_{gi} as well as c_{di} .

\mathbf{S} is the matrix corresponding to the sensitivities of the constraints, but it also includes in its first row the power balance condition, thus its dimension is $m + n$ by $n_s + 1$.

\mathbf{s} is a vector with a zero in its first position and the values of the right hand sides in the constraint equations in the remaining positions.

The equilibrium point of these equations is obtained from the solution of the following purely algebraic equations:

$$\begin{bmatrix} \mathbf{C} & \mathbf{S}^t \\ \mathbf{S} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{P} \\ \Lambda \end{bmatrix} = - \begin{bmatrix} \mathbf{b} \\ \mathbf{s} \end{bmatrix} \quad (44)$$

Key observations about these equations are:

- The main effect of congestion is to result in at least a partial separation of markets: the solution no longer occurs at equal marginal costs for all producers⁹.
- Which and how many equations join the algebraic constraints depends on the nature and degree of congestion.
- The lagrange multipliers μ are not unique. They can all be altered by adding an arbitrary constant to all multipliers and the solution is unaltered.

The dynamic equations for this problem are of reduced order. Reduction of the DAE problem to a purely differential equation can be done by first eliminating all the $n_s + 1$ lagrange multipliers, and then replacing $n_s + 1$ redundant state variables in terms of a reduced set of $n + m - n_s - 1$ of non-redundant variables,

⁹Whether the prices are made explicitly by congestion pricing as in [9] or whether the congestion pricing develops implicitly as a result of trade coordination is irrelevant. Without price separation as a result of congestion an equilibrium is not possible.

obtaining in the end a set of $n + m - n_s - 1$ purely differential equations. To do this, the equations above are organized as follows:

$$\begin{bmatrix} \mathbf{T}_1 & 0 & \mathbf{S}_1^t \\ 0 & \mathbf{T}_2 & \mathbf{S}_2^t \\ \mathbf{S}_1 & \mathbf{S}_2 & 0 \end{bmatrix} \begin{bmatrix} \dot{\mathbf{P}}_1 \\ \dot{\mathbf{P}}_2 \\ \dot{\Lambda} \end{bmatrix} = \begin{bmatrix} \mathbf{C}_1 & 0 & \mathbf{S}_1^t \\ 0 & \mathbf{C}_2 & \mathbf{S}_2^t \\ \mathbf{S}_1 & \mathbf{S}_2 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{P}_1 \\ \mathbf{P}_2 \\ \Lambda \end{bmatrix} \quad (45)$$

where \mathbf{S}_1 corresponds to a subset of $n_s + 1$ by $n_s + 1$ of \mathbf{S} , and \mathbf{P}_1 corresponds to a subset n_s of \mathbf{P} denoting the redundant power variables. The classification of \mathbf{P} into redundant and nonredundant states is irrelevant, and is subject only to the condition that \mathbf{S}_1 be nonsingular. Reduction and elimination of Λ and \mathbf{P}_1 yields the following reduced purely differential equations:

$$[\mathbf{T}_2 + \mathbf{S}_2^t \mathbf{S}_1^{-t} \mathbf{T}_1 \mathbf{S}_1^{-1} \mathbf{S}_2] \dot{\mathbf{P}}_2 = [\mathbf{C}_2 + \mathbf{S}_2^t \mathbf{S}_1^{-t} \mathbf{C}_1 \mathbf{S}_1^{-1} \mathbf{S}_2] \mathbf{P}_2 \quad (46)$$

where \mathbf{S}_1^{-t} is the transposed inverse of \mathbf{S}_1 .

The stability characteristics of the congested problem depend on the location of the eigenvalues for this reduced dynamic system. Observations from this equation include:

- The effect of congestion on the dynamics is to reduce the order of the dynamic interactions but to render results less intuitive.
- The dynamics of the congestion pricing mechanism have not been considered.

Thus, congestion simply alters the number and nature of the already present algebraic constraints under which the market must operate.

8 Computational comments

This report has used full matrix techniques and explicit reduction of algebraic equations to obtain conditions that lead to stable behavior. Furthermore, the power balance equation has ignored all network effects, including losses. All the methods and results in this paper can be extended by replacing the single equation denoting power balance by the complete set of power flow equations for the problem. In addition, all the eigenvalue computations can take place in the unreduced problem. It is not necessary to explicitly reduce the matrix problems to lower order in order to determine the eigenvalues of the reduced system. There are quite effective methods and techniques able to deal with generalized unreduced DAE eigenvalue problems[10].

9 The numerical examples

This section presents numerical examples to illustrate the characteristics of the various models and assumptions.

9.1 One-supplier one-consumer example

The first example considers one supplier and one consumer, both elastic. A variety of parameters and coefficients are considered. These are illustrated in Table 1. From this table, the following observations can be made:

- Changes in the linear cost coefficients b affect the location of the operating point. It is seen that changes in b result in changes in P_g and P_d . These changes also affect the cost λ .
- Changes to b do not affect the dynamic characteristics of the problem.
- Modifications to the time constants τ_g and τ_d affect the dynamic characteristics of the market, but do not alter the operating point.
- Changes to c_g and/or c_d affect both the location of the market equilibrium and the dynamic characteristics of the market.

These conclusions underscore the significance of the quadratic cost coefficient (c) of suppliers in determining market equilibria characteristics.

Table 1: Equilibrium and dynamic characteristics of a one supplier one demand system without congestion. Changes relative to the base case are highlighted.

Supplier data			Consumer data			SS solution			
τ_g	c_g	b_g	τ_d	c_d	b_d	P_g	P_d	λ	Eig.
0.3	0.5	2	0.2	-0.5	10	8.0	8.0	6.00	-2.00
0.3	0.5	4	0.2	-0.5	9	5.0	5.0	6.50	-2.00
0.2	0.5	2	0.2	-0.5	10	8.0	8.0	6.00	-2.50
0.3	0.5	2	0.2	-0.2	10	11.4	11.4	7.71	-1.40
0.3	0.5	2	0.2	-2.0	10	3.2	3.2	3.60	-5.00

9.2 Two-supplier inelastic demand example

The next example is a two-supplier case with fixed (completely inelastic) demand. Table 2 illustrates the values of the steady-state solution point as well as the dynamic characteristics of the market. As before, only one eigenvalue is needed to characterize this case. The following observations can be drawn from the example:

- If one supplier has economies of scale and the other does not, the issue of stability depends on the values of certain other parameters.
- If the market is stable for one set of values of τ (market delays), it will be stable for any other value of τ .

Table 2: Equilibrium and dynamic characteristics of a two-supplier fixed-demand system without congestion. Changes relative to the base case are highlighted.

Input data				Equilibrium		
τ_g	c_g	b_g	P_D	P_g	λ	Eig.
0.3	0.5	2.0	10.0	1.43	2.71	-1.40
0.2	0.2	1.0		8.57		
0.3	0.5	2.0	8.0	0.86	2.43	-1.40
0.2	0.2	1.0		7.14		
0.3	0.5	2.0	10.0	5.71	4.86	-1.40
0.2	0.2	4.0		4.29		
0.3	0.5	2.0	10.0	3.33	3.67	-0.60
0.2	-0.2	5.0		6.67		
0.3	-0.3	3.5	10.0	5.00	2.00	0.20
0.2	0.2	1.0		5.00		

This example illustrates the significance of the quadratic cost coefficient (the elasticity coefficients) for the determination of stability of markets: changes in the elasticity coefficients directly affect the solution eigenvalues and can even make the market unstable. The last case has a positive eigenvalue.

9.3 Multi-supplier multi-consumer examples

The next example considers a variety of increasingly larger order systems, with more generators and loads. The case is illustrated in Table 3.

These examples verify the earlier analysis that indicates that, if economies of scale exist, unstable markets can occur. The following inferences can be drawn from this example:

- The steady state operating point depends on the linear cost coefficients b . Changing the costs of power for the various producers has an effect on the ultimate cost as well as the ultimate production schedules.
- However, the steady state operating point does *not* depend on the time constants τ or the quadratic cost coefficients c_g or c_d .
- The stability of the operating point depends on the quadratic coefficients c_g or c_d .

9.4 Congestion examples

The next set of examples considers the effect of congestion on both the equilibrium point for the market and the stability of this equilibrium point. The congestion example is illustrated in Table 4. This table illustrates a three-supplier

Table 3: A two-supplier one-consumer case, and a 3-supplier two-consumer case for various time constants and cost coefficients. The last two cases exhibit economies of scale, but only the last case results in an unstable market (see text for details). Interesting numbers within the table are indicated boldface.

Supplier data			Consumer data			SS solution			
τ_g	c_g	b_g	τ_d	c_d	b_d	P_g	P_d	λ	Eig.
0.3	0.5	2.0	0.20	-0.50	10.0	2.44	13.56	3.22	-1.34
0.2	0.2	1.0				11.11			-2.10
0.3	0.5	3.0	0.20	-0.50	9.0	0.44	11.56	3.22	-1.34
0.2	0.2	1.0				11.11			-2.10
0.3	0.5	2.0	0.20	-0.50	10.0	2.52	13.48	3.26	-1.24
0.2	0.2	1.0	0.25	-0.60	8.0	11.31	7.90		-1.85
0.1	0.3	1.0				7.54			-2.74
									-2.44
0.3	0.5	2.0	0.20	-0.50	10.0	4.67	11.33	4.33	-0.04
0.2	-0.1	4.5	0.25	-0.60	8.0	1.67	6.11		-1.83
0.1	0.3	1.0				11.11			-2.74
									-2.44
0.3	0.5	2.0	0.20	-0.50	10.0	3.62	12.38	3.81	0.50
0.2	-0.1	5.0	0.25	-0.60	8.0	11.92	6.99		-0.93
0.1	-0.05	4.0				3.84			-1.95
									-2.45

two-consumer case with one, two and three congestion constraints active. In addition to the usual power balance condition, the congestion constraints in this hypothetical example are characterized by the following conditions:

$$\begin{bmatrix} 0.1 & -0.1 & 0 & 0.1 & -0.1 \\ 0.2 & 0 & 0.3 & -0.1 & -0.1 \\ 0.1 & 0.1 & 0.1 & 0 & -0.4 \end{bmatrix} \begin{bmatrix} P_{g1} \\ P_{g2} \\ P_{g3} \\ P_{d1} \\ P_{d2} \end{bmatrix} = \begin{bmatrix} 1.00 \\ 0.50 \\ 1.75 \end{bmatrix} \quad (47)$$

Four cases of increasing congestion are considered. The first case ($n_s = 0$) ignores congestion constraints. The second case ($n_s = 1$) considers only the first row in the constraint equations 47. The next case ($n_s = 2$) considers two congested flows (the first two rows in equation 47). The last case ($n_s = 3$) considers all rows.

The following are observations from this case:

- As the number of active congestion constraints increases, the order of the system dynamics (the number of eigenvalues) decreases.

Table 4: A three-supplier two-consumer case with increasing degrees of congestion.

n_s	Supplier data			Consumer data			SS solution			Eig.
	τ_g	c_g	b_g	τ_d	c_d	b_d	P_g	P_d	Λ	
0	0.1	0.3	1.0	0.20	-0.50	10.0	7.54	13.48	3.26	-1.24
	0.3	0.5	2.0	0.25	-0.60	8.0	2.52	7.90		-1.85
	0.2	0.2	1.0				11.31			-2.74 -2.44
1	0.1	0.3	1.0	0.20	-0.50	10.0	0.40	8.53	3.43	-1.24
	0.3	0.5	2.0	0.25	-0.60	8.0	7.47	11.47	23.07	-2.03
	0.2	0.2	1.0				12.13			-2.58
2	0.1	0.3	1.0	0.20	-0.50	10.0	1.89	7.68	6.27	-2.00
	0.3	0.5	2.0	0.25	-0.60	8.0	11.52	8.05	14.96	-2.42
	0.2	0.2	1.0				2.31		16.01	
3	0.1	0.3	1.0	0.20	-0.50	10.0	2.30	7.56	6.49	-2.05
	0.3	0.5	2.0	0.25	-0.60	8.0	11.51	8.35	13.51	
	0.2	0.2	1.0				2.10		16.51	1.18

- Congestion affects the number of eigenvalues. However, if the case is stable without congestion, it remains stable after the onset of congestion.
- Congestion affects the operating point significantly.
- Negative values for either P_g or P_d can result. Negative values would indicate that the corresponding output should become fixed at zero, thereby adding another constraint and reducing the dynamic order of the system further.
- The operating cost increases as congestion increases.
- Although not illustrated in this example, it is possible for values of the lagrange multiplier entries in Λ to change sign. This is an indication that the corresponding constraint has become superseded (irrelevant) and must be removed (similar to the familiar Kuhn-Tucker conditions).

9.5 Energy imbalance examples

The next example considers the presence of energy imbalance, along with a generator-based pricing mechanism to relieve the imbalance. The example is illustrated in Table 5, and it corresponds to a simple case of one-supplier, one-consumer, and a simple energy imbalance management system.

The main observations from these results are:

Table 5: The stability properties associated with energy imbalance for a one supplier one consumer system.

Supplier data			Consumer data			Other		SS solution			Eig.
τ_g	c_g	b_g	τ_d	c_d	b_d	τ_λ	k	P_g	P_d	λ	
0.2	0.1	2	0.1	-0.2	10	100	0.1	26.67	26.67	4.67	-0.16 ± 0.68 -0.15 -2.02
0.2	0.1	2	0.1	-0.2	10	10	0.1	26.67	26.67	4.67	$+0.17 \pm 1.01$ -0.65 -2.19
0.2	0.1	2	0.1	-0.2	10	100	0.01	26.67	26.67	4.67	$+0.04 \pm 0.36$ -0.56 -2.02

- The market can develop an oscillatory instability when mechanisms to reduce energy imbalance are put in place. That is, the market will cycle between conditions of insufficient power and conditions of excess power, where the amplitude of the oscillations increases with time.
- The resulting instability can be caused by too rapid a response of the price correction mechanism intended to maintain system frequency.
- The term k in equation 34 represents a stabilizing corrective price signal. This signal is required for stability of the market. Without it (i.e., if $k = 0$), the market is unstable. The value of k required to stabilize the market depends on the value of τ_λ , the speed with which prices are adjusted when an energy imbalance occurs.
- Neither parameter affects the steady state market equilibrium point, only its stability properties.

The implication of all this is that care must be exercised in the design of markets: the ability of markets to respond too rapidly to power insufficiencies or excesses can give rise to instability, as can the case where the market is designed to be too lax and not required to maintaining adequate balance between production and consumption at all times.

10 Conclusions

Stable well-behaved markets for electric power require consideration of the dynamics of those markets. Ignoring the dynamics of markets for electric power can result in markets that are economically unstable exhibiting great price volatility. This, in turn, can lead to energy systems that are less secure with frequent

outages and emergencies, as security constraints are not properly enforced in a timely fashion. It can also lead to conditions where monopolistic behavior becomes possible, even with multiple competing market suppliers. All this can be avoided by the proper design of the market and its parameters. Tools for better market design include the introduction of price signals associated with specific ancillary services (frequency regulation and load imbalance services are the main two considered in this report), the introduction of deliberate time delays in certain actions in the system, and the establishment of effective means for congestion management and prevention.

Since tradability and simplicity of trades are essential for markets to operate properly, it is suggested that a careful and periodic study of the changing dynamics of power markets will be a perpetual requirement to assure that inconsistencies and instabilities do not develop.

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