Inclusion of Price Dependent Load Models in the Optimal Power Flow

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Abstract

In this paper we investigate the inclusion of price-dependent loads into the traditional Optimal Power Flow algorithm. The development of the model will be based on the solution of the OPF using an objective function for maximization of social welfare. The paper will show that a traditional OPF algorithm that minimizes supplier costs can be modified to solve the social welfare maximization problem by including price-dependent load models. This modification to the standard OPF is intuitive and very simple. We will show this modified OPF formulation facilitates simulation of a spot market for electricity. While the development in this paper will account for both real and reactive power supply and consumption, the examples in the paper will concentrate on real power markets. The algorithm will be demonstrated on a range of practical examples, including several small systems, and on a system with over a hundred buses. The impact of such price dependent loads on congestion and bus marginal costs is highlighted.

Key Words

Optimal Power Flow (OPF), Real-Time Pricing, Price Dependent Loads, Real and Reactive Power

1. Introduction

Over the last thirty years or so the optimal power flow (OPF) algorithm has been an active area of research. The OPF is defined as a static, nonlinear optimization problem in which certain control variables are adjusted to minimize an objective function, while satisfying physical and operational constraints. Typically the objective function has been to either minimize the cost of generation, or to minimize system losses. Available controls have usually been power system devices, such as generator real and reactive power outputs, real power transactions between operating areas, transformer tap or phase positions, and switched shunt devices. Customer load has usually not been an explicit control device, except in the extreme case of "load shedding" in which the load is involuntarily disconnected.

The absence of load as a control in the OPF has been due, for the most part, to the inability of the operating utility to directly control or indirectly control the load. The present-day flat and time-of-use rate structures have provided no opportunity for price based control of most loads, with interruptible contracts and direct load management two possible exceptions.

However over the last ten to fifteen years there has been a growing movement towards providing customers with more price feedback through an electric spot market. Much of the theory for such a market is described in [1], with the definition of a spot price given as one in which customers are charged the marginal cost of providing electricity to their point of service (that is, their node or bus). Other papers have addressed the issues of power system spot markets as well [2],[3]. A key advantage of nodal spot prices is they provide a more economic approach to pricing with a result of improved transmission efficiency. In such a market, customer load is assumed to vary in response to changing prices according to its demand curve. That is, load becomes price dependent and hence a potential OPF control.

In this paper we investigate the inclusion of such price dependent loads in the OPF. While the inclusion of such price dependent loads has been done [4], here we provide a more formal argument for their inclusion and show how they can easily be added to a standard OPF. The OPF problem has been solved using a variety of different techniques. Here we employ the Newton method approach [5].

Section 2 provides an overview of the notation used throughout the paper. Section 3 provides background on the standard OPF that minimizes supplier costs. Section 4 introduces the objective function of maximizing social welfare. Section 5 proves and explains the validity of an alternate approach to maximizing social welfare. Section 6 further explains the significance of the consumer demand function. Section 7 shows the ease in which the price dependent loads may be added to an existing OPF algorithm. Section 8 shows several simulations using the new OPF algorithm that maximizes social welfare.

Section 9 introduces the possibility of a further simplification to the technique aiding in the simulation of a two-sided demand and supply market. Finally, Section 10 provides the conclusion.

2. Notation

General conventions on notation for this paper

- All vector and matrix variables are in bold.
- All vectors are column vectors.
- Subscript p and subscript q signify a relation to real and reactive power respectively.

Variable Definitions

$\mathbf{x}$ = state variables and other controls (e.g. tap ratios)

$\mathbf{s} = \left[\mathbf{s}_p^T \quad \mathbf{s}_q^T\right]^T =$ the supply vector

$\mathbf{d} = \left[\mathbf{d}_p^T \quad \mathbf{d}_q^T\right]^T =$ the demand vector

$\hat{\mathbf{s}} = \left[\hat{\mathbf{s}}_p^T \quad \hat{\mathbf{s}}_q^T\right]^T =$ augmented supply vector including zeros where no suppliers exists

$\hat{\mathbf{d}} = \left[\hat{\mathbf{d}}_p^T \quad \hat{\mathbf{d}}_q^T\right]^T =$ augmented demand vector including zeros where no loads exist

$C(\mathbf{s}) = C(\mathbf{s}_p, \mathbf{s}_q) = \sum_{\text{all suppliers}} C_k(\mathbf{s}_p, \mathbf{s}_q)$ = Suppliers' Cost

$B(\mathbf{d}) = B(\mathbf{d}_p, \mathbf{d}_q) = \sum_{\text{all consumers}} B_k(\mathbf{d}_p, \mathbf{d}_q)$ = Consumers' Benefit

$$\mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d}) = \begin{bmatrix} \hat{\mathbf{h}}(\mathbf{x}) - \hat{\mathbf{s}} + \hat{\mathbf{d}} \\ \overline{\mathbf{h}}(\mathbf{x}) \end{bmatrix} = \begin{bmatrix} \mathbf{h}_p(\mathbf{x},\mathbf{s}_p,\mathbf{d}_p) \\ \mathbf{h}_q(\mathbf{x},\mathbf{s}_q,\mathbf{d}_q) \\ \overline{\mathbf{h}}(\mathbf{x}) \end{bmatrix} = \text{equality constraints}$$

$\mathbf{h}_p(\mathbf{x},\mathbf{s}_p,\mathbf{d}_p) = \hat{\mathbf{h}}_p(\mathbf{x}) - \hat{\mathbf{s}}_p + \hat{\mathbf{d}}_p$ = real power flow equations. See Appendix for more detail.

$\mathbf{h}_q(\mathbf{x},\mathbf{s}_q,\mathbf{d}_q) = \hat{\mathbf{h}}_q(\mathbf{x}) - \hat{\mathbf{s}}_q + \hat{\mathbf{d}}_q$ = reactive power flow equations. See Appendix for more detail.

$$\mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d}) = \begin{bmatrix} \mathbf{s}_{\min} - \mathbf{s} \\ \mathbf{s} - \mathbf{s}_{\max} \\ \mathbf{d}_{\min} - \mathbf{d} \\ \mathbf{d} - \mathbf{d}_{\max} \\ \overline{\mathbf{g}}(\mathbf{x}) \end{bmatrix} = \text{inequality constraints}$$

$\mathbf{f}(\mathbf{d},\mathbf{p}_d)$ = additional equation for consumer demand.

L, $\overline{L}$, $\tilde{L}$ = Lagrange functions

$\boldsymbol{\lambda} = \left[\boldsymbol{\lambda}_h^T \quad \boldsymbol{\lambda}_g^T \quad \boldsymbol{\lambda}_f^T\right]^T$ = Lagrange multiplier vector

$\boldsymbol{\lambda}_h = \left[\boldsymbol{\lambda}_{\hat{h}}^T \quad \boldsymbol{\lambda}_{\overline{h}}^T\right]^T = \left[\boldsymbol{\lambda}_{\hat{h}p}^T \quad \boldsymbol{\lambda}_{\hat{h}q}^T \quad \boldsymbol{\lambda}_{\overline{h}}^T\right]^T$ = Lagrange multiplier vector for power flow equations and other equality constraints.

$\boldsymbol{\lambda}_g = \left[\boldsymbol{\lambda}_{gs\min}^T \quad \boldsymbol{\lambda}_{gs\max}^T \quad \boldsymbol{\lambda}_{gd\min}^T \quad \boldsymbol{\lambda}_{gd\max}^T \quad \boldsymbol{\lambda}_{\overline{g}}^T\right]^T$ = Lagrange multiplier vector for inequality constraints

$\tilde{\boldsymbol{\lambda}}_{\hat{h}s} = \left[\tilde{\boldsymbol{\lambda}}_{\hat{h}sp}^T \quad \tilde{\boldsymbol{\lambda}}_{\hat{h}sq}^T\right]^T$ = reduced Lagrange multiplier vector including only entries for power flow equations which include a supply of real or reactive power.

$\tilde{\boldsymbol{\lambda}}_{\hat{h}d} = \left[\tilde{\boldsymbol{\lambda}}_{\hat{h}dp}^T \quad \tilde{\boldsymbol{\lambda}}_{\hat{h}dq}^T\right]^T$ = reduced Lagrange multiplier vector including only entries for power flow equations which include a demand of real or reactive power

$\boldsymbol{\lambda}_f = \left[\boldsymbol{\lambda}_{fp}^T \quad \boldsymbol{\lambda}_{fq}^T\right]^T$ = Lagrange multiplier vector for additional constraints

$\mathbf{p} = \left[\mathbf{p}_s^T \;\vdots\; \mathbf{p}_d^T\right]^T = \left[\mathbf{p}_{sp}^T \quad \mathbf{p}_{sq}^T \;\vdots\; \mathbf{p}_{dp}^T \quad \mathbf{p}_{dq}^T\right]^T$ = price vector for variable suppliers and variable consumers. (includes real and reactive prices)

$\mathbf{D}(\bullet)$ = is the functional inverse of $\dfrac{\partial B(\bullet)}{\partial \mathbf{d}}$. At optimal solution this is the consumer demand function.

$\mathbf{S}(\bullet)$ = is the functional inverse of $\dfrac{\partial C(\bullet)}{\partial \mathbf{s}}$. At optimal solution this is the supplier supply function.

3. Standard OPF Formulation

For background, the standard OPF with the objective of minimizing generation costs is described in this section of the paper. As mentioned in the introduction, the consumer demand is not typically a variable in this problem. In order for this development to match later equations, we will maximize the negative of the costs instead of minimizing the costs.

$$\begin{aligned} &\max_{\mathbf{s},\mathbf{x}} \quad -C(\mathbf{s}) \\ &\text{s.t.} \quad \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d}) = \begin{bmatrix} \hat{\mathbf{h}}(\mathbf{x}) - \hat{\mathbf{s}} + \hat{\mathbf{d}} \\ \overline{\mathbf{h}}(\mathbf{x}) \end{bmatrix} = \mathbf{0} \\ &\qquad \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d}) = \begin{bmatrix} \mathbf{s}_{\min} - \mathbf{s} \\ \mathbf{s} - \mathbf{s}_{\max} \\ \mathbf{d}_{\min} - \mathbf{d} \\ \mathbf{d} - \mathbf{d}_{\max} \\ \overline{\mathbf{g}}(\mathbf{x}) \end{bmatrix} \le \mathbf{0} \end{aligned} \tag{3.1}$$

To solve this nonlinear program, form the Lagrange function for it.

$$\begin{aligned} \overline{L} &= -C(\mathbf{s}) + \boldsymbol{\lambda}_h^T \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d}) + \boldsymbol{\lambda}_g^T \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d}) \\ &= \begin{pmatrix} -C(\mathbf{s}) + \boldsymbol{\lambda}_{\hat{h}}^T[\hat{\mathbf{h}}(\mathbf{x}) - \hat{\mathbf{s}} + \hat{\mathbf{d}}] + \boldsymbol{\lambda}_{\overline{h}}^T[\overline{\mathbf{h}}(\mathbf{x})] \\ + \boldsymbol{\lambda}_{gs\min}^T[\mathbf{s}_{\min} - \mathbf{s}] + \boldsymbol{\lambda}_{gs\max}^T[\mathbf{s} - \mathbf{s}_{\max}] \\ + \boldsymbol{\lambda}_{gd\min}^T[\mathbf{d}_{\min} - \mathbf{d}] + \boldsymbol{\lambda}_{gd\max}^T[\mathbf{d} - \mathbf{d}_{\max}] + \boldsymbol{\lambda}_{\overline{g}}^T \overline{\mathbf{g}}(\mathbf{x}) \end{pmatrix} \end{aligned} \tag{3.2}$$

The problem can then be determined by solving for the Kuhn-Tucker conditions [6].

Stationarity Conditions

$$\frac{\partial \bar{L}}{\partial \mathbf{x}} = \boldsymbol{\lambda}_h{}^T \frac{\partial \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d})}{\partial \mathbf{x}} + \boldsymbol{\lambda}_g{}^T \frac{\partial \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d})}{\partial \mathbf{x}} = \mathbf{0}$$
$$\frac{\partial \bar{L}}{\partial \mathbf{s}} = -\frac{\partial C(\mathbf{s})}{\partial \mathbf{s}} - \tilde{\boldsymbol{\lambda}}_{\hat{h}s} - \boldsymbol{\lambda}_{gs\min} + \boldsymbol{\lambda}_{gs\max} = \mathbf{0} \qquad (3.3)$$
$$\frac{\partial \bar{L}}{\partial \boldsymbol{\lambda}_h} = \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d}) = \mathbf{0}$$

Complementary Slackness Conditions

$$\boldsymbol{\lambda}_g^T \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d}) = \mathbf{0} \quad ; \quad \boldsymbol{\lambda}_g \geq \mathbf{0}$$

4. Maximizing Social Welfare

In order to maximize social welfare within a power system, the objective function for the OPF described in Section 3 need only be modified to include the consumer benefit function $B(\mathbf{d})$.

$$\max_{\mathbf{x},\mathbf{s},\mathbf{d}} \quad B(\mathbf{d}) - C(\mathbf{s})$$
$$\text{s.t.} \quad \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d}) = \begin{bmatrix} \hat{\mathbf{h}}(\mathbf{x}) - \hat{\mathbf{s}} + \hat{\mathbf{d}} \\ \bar{\mathbf{h}}(\mathbf{x}) \end{bmatrix} = \mathbf{0}$$
$$\mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d}) = \begin{bmatrix} \mathbf{s}_{\min} - \mathbf{s} \\ \mathbf{s} - \mathbf{s}_{\max} \\ \mathbf{d}_{\min} - \mathbf{d} \\ \mathbf{d} - \mathbf{d}_{\max} \\ \bar{\mathbf{g}}(\mathbf{x}) \end{bmatrix} \leq \mathbf{0} \qquad (4.1)$$

Again, to solve this, form a Lagrange function as follows

$$L = B(\mathbf{d}) - C(\mathbf{s}) + \boldsymbol{\lambda}_h^T \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d}) + \boldsymbol{\lambda}_g^T \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d})$$
$$= \begin{pmatrix} B(\mathbf{d}) - C(\mathbf{s}) + \boldsymbol{\lambda}_{\hat{h}}^T[\hat{\mathbf{h}}(\mathbf{x}) - \hat{\mathbf{s}} + \hat{\mathbf{d}}] + \boldsymbol{\lambda}_{\bar{h}}^T[\bar{\mathbf{h}}(\mathbf{x})] \\ + \boldsymbol{\lambda}_{gs\min}^T[\mathbf{s}_{\min} - \mathbf{s}] + \boldsymbol{\lambda}_{gs\max}^T[\mathbf{s} - \mathbf{s}_{\max}] \\ + \boldsymbol{\lambda}_{gd\min}^T[\mathbf{d}_{\min} - \mathbf{d}] + \boldsymbol{\lambda}_{gd\max}^T[\mathbf{d} - \mathbf{d}_{\max}] + \boldsymbol{\lambda}_{\bar{g}}^T \bar{\mathbf{g}}(\mathbf{x}) \end{pmatrix} \qquad (4.2)$$

The maximization problem can then be determined by solving for the Kuhn-Tucker conditions.

Stationarity Conditions

$$\frac{\partial L}{\partial \mathbf{x}} = \boldsymbol{\lambda}_h{}^T \frac{\partial \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d})}{\partial \mathbf{x}} + \boldsymbol{\lambda}_g{}^T \frac{\partial \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d})}{\partial \mathbf{x}} = \mathbf{0}$$
$$\frac{\partial L}{\partial \mathbf{s}} = -\frac{\partial C(\mathbf{s})}{\partial \mathbf{s}} - \tilde{\boldsymbol{\lambda}}_{\hat{h}s} - \boldsymbol{\lambda}_{gs\min} + \boldsymbol{\lambda}_{gs\max} = \mathbf{0}$$
$$\frac{\partial L}{\partial \mathbf{d}} = \frac{\partial B(\mathbf{d})}{\partial \mathbf{d}} + \tilde{\boldsymbol{\lambda}}_{\hat{h}d} - \boldsymbol{\lambda}_{gd\min} + \boldsymbol{\lambda}_{gd\max} = \mathbf{0} \qquad (4.3)$$
$$\frac{\partial L}{\partial \boldsymbol{\lambda}_h} = \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d}) = \mathbf{0}$$

Complementary Slackness Conditions

$$\boldsymbol{\lambda}_g^T \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d}) = \mathbf{0} \quad ; \quad \boldsymbol{\lambda}_g \geq \mathbf{0}$$

The solution to equations (4.3) would then constitute a maximization of the social welfare for a power market.

5. Alternative Approach

While the approach for determining the maximization of social welfare shown in Section 4 could be implemented, it would require the addition of variables and augmentation of the control variables. It would be preferable if only minor changes need be made to the standard OPF formulations. In order to achieve this, consider solving the family of nonlinear programs parameterized by $\mathbf{p}_d$.

$$\max_{\mathbf{x},\mathbf{s},\mathbf{d}} \quad -C(\mathbf{s})$$
$$\mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d}) = \begin{bmatrix} \hat{\mathbf{h}}(\mathbf{x}) - \hat{\mathbf{s}} + \hat{\mathbf{d}} \\ \bar{\mathbf{h}}(\mathbf{x}) \end{bmatrix} = \mathbf{0}$$
$$\text{s.t.} \quad \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d}) = \begin{bmatrix} \mathbf{s}_{\min} - \mathbf{s} \\ \mathbf{s} - \mathbf{s}_{\max} \\ \mathbf{d}_{\min} - \mathbf{d} \\ \mathbf{d} - \mathbf{d}_{\max} \\ \bar{\mathbf{g}}(\mathbf{x}) \end{bmatrix} \leq \mathbf{0} \qquad (5.1)$$
$$\mathbf{f}(\mathbf{d},\mathbf{p}_d) = \mathbf{d} - \mathbf{D}(\mathbf{p}_d) = \mathbf{0}$$

where the function $\mathbf{D}(\bullet)$ is the functional inverse of $\frac{\partial B(\bullet)}{\partial \mathbf{d}}$. In other words, $\mathbf{p}_d - \frac{\partial B(\mathbf{D}(\mathbf{p}_d))}{\partial \mathbf{d}} = 0 \quad \forall \mathbf{p}_d$.

Again, to solve this, form a Lagrange function as follows

$$\tilde{L} = -C(\mathbf{s}) + \boldsymbol{\lambda}_h^T \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d}) + \boldsymbol{\lambda}_g^T \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d}) + \boldsymbol{\lambda}_f^T \mathbf{f}(\mathbf{d},\mathbf{p}_d)$$
$$= \begin{pmatrix} -C(\mathbf{s}) + \boldsymbol{\lambda}_{\hat{h}}^T[\hat{\mathbf{h}}(\mathbf{x}) - \hat{\mathbf{s}} + \hat{\mathbf{d}}] + \boldsymbol{\lambda}_{\bar{h}}^T[\overline{\mathbf{h}}(\mathbf{x})] + \boldsymbol{\lambda}_{gs\min}^T[\mathbf{s}_{\min} - \mathbf{s}] \\ + \boldsymbol{\lambda}_{gs\max}^T[\mathbf{s} - \mathbf{s}_{\max}] + \boldsymbol{\lambda}_{gd\min}^T[\mathbf{d}_{\min} - \mathbf{d}] + \boldsymbol{\lambda}_{gd\max}^T[\mathbf{d} - \mathbf{d}_{\max}] \\ + \boldsymbol{\lambda}_{\bar{g}}^T \overline{\mathbf{g}}(\mathbf{x}) + \boldsymbol{\lambda}_f^T[\mathbf{d} - \mathbf{D}(\mathbf{p}_d)] \end{pmatrix} \quad (5.2)$$

This maximization problem can then be determined by solving for the Kuhn-Tucker conditions.

Stationarity Conditions

$$\begin{aligned} \frac{\partial \tilde{L}}{\partial \mathbf{x}} &= \boldsymbol{\lambda}_h{}^T \frac{\partial \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d})}{\partial \mathbf{x}} + \boldsymbol{\lambda}_g{}^T \frac{\partial \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d})}{\partial \mathbf{x}} = \mathbf{0} \\ \frac{\partial \tilde{L}}{\partial \mathbf{s}} &= -\frac{\partial C(\mathbf{s})}{\partial \mathbf{s}} - \tilde{\boldsymbol{\lambda}}_{\hat{h}s} - \boldsymbol{\lambda}_{gs\min} + \boldsymbol{\lambda}_{gs\max} = \mathbf{0} \\ \frac{\partial \tilde{L}}{\partial \mathbf{d}} &= +\tilde{\boldsymbol{\lambda}}_{\hat{h}d} - \boldsymbol{\lambda}_{gd\min} + \boldsymbol{\lambda}_{gd\max} + \boldsymbol{\lambda}_f = \mathbf{0} \\ \frac{\partial \tilde{L}}{\partial \boldsymbol{\lambda}_h} &= \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d}) = \mathbf{0} \\ \frac{\partial \tilde{L}}{\partial \boldsymbol{\lambda}_f} &= \mathbf{f}(\mathbf{d},\mathbf{p}_d) = \mathbf{d} - \mathbf{D}(\mathbf{p}_d) = \mathbf{0} \end{aligned} \quad (5.3)$$

Complementary Slackness Conditions

$$\boldsymbol{\lambda}_g^T \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d}) = \mathbf{0} \quad ; \quad \boldsymbol{\lambda}_g \geq \mathbf{0}$$

Now impose an "after the fact" equation of $\boldsymbol{\lambda}_f = \mathbf{p}_d$ and the necessary conditions become

$$\begin{aligned} \boldsymbol{\lambda}_h{}^T \frac{\partial \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d})}{\partial \mathbf{x}} + \boldsymbol{\lambda}_g{}^T \frac{\partial \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d})}{\partial \mathbf{x}} &= \mathbf{0} \\ -\frac{\partial C(\mathbf{s})}{\partial \mathbf{s}} - \tilde{\boldsymbol{\lambda}}_{\hat{h}s} - \boldsymbol{\lambda}_{gs\min} + \boldsymbol{\lambda}_{gs\max} &= \mathbf{0} \\ +\tilde{\boldsymbol{\lambda}}_{\hat{h}d} - \boldsymbol{\lambda}_{gd\min} + \boldsymbol{\lambda}_{gd\max} + \boldsymbol{\lambda}_f &= \mathbf{0} \\ \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d}) &= \mathbf{0} \\ \mathbf{d} - \mathbf{D}(\boldsymbol{\lambda}_f) &= \mathbf{0} \\ \boldsymbol{\lambda}_g^T \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d}) = \mathbf{0} \quad ; \quad \boldsymbol{\lambda}_g &\geq \mathbf{0} \end{aligned} \quad (5.4)$$

Then using the definition $\mathbf{D}(\bullet)$ as the functional inverse of $\frac{\partial B(\bullet)}{\partial \mathbf{d}}$, $\mathbf{d} - \mathbf{D}(\boldsymbol{\lambda}_f) = \mathbf{0}$ implies $\boldsymbol{\lambda}_f - \frac{\partial B(\mathbf{d})}{\partial \mathbf{d}} = \mathbf{0}$. Therefore the necessary conditions may be written as

$$\begin{aligned} \boldsymbol{\lambda}_h{}^T \frac{\partial \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d})}{\partial \mathbf{x}} + \boldsymbol{\lambda}_g{}^T \frac{\partial \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d})}{\partial \mathbf{x}} &= \mathbf{0} \\ -\frac{\partial C(\mathbf{s})}{\partial \mathbf{s}} - \tilde{\boldsymbol{\lambda}}_{\hat{h}s} - \boldsymbol{\lambda}_{gs\min} + \boldsymbol{\lambda}_{gs\max} &= \mathbf{0} \\ \frac{\partial B(\mathbf{d})}{\partial \mathbf{d}} + \tilde{\boldsymbol{\lambda}}_{\hat{h}d} - \boldsymbol{\lambda}_{gd\min} + \boldsymbol{\lambda}_{gd\max} &= \mathbf{0} \\ \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{d}) &= \mathbf{0} \\ \boldsymbol{\lambda}_g^T \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{d}) = \mathbf{0} \quad ; \quad \boldsymbol{\lambda}_g &\geq \mathbf{0} \end{aligned} \quad (5.5)$$

These are the same necessary conditions as equations (4.3) which result in maximization of social welfare. Therefore, this nonlinear program of equations (5.1) yields the same solution as maximization of social welfare.

Consider combining the third and fifth of equations (5.4) to form

$$\mathbf{d} - \mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max}) = \mathbf{0}\,. \quad (5.6)$$

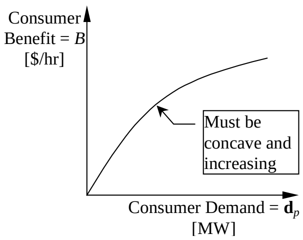
Now substitute equation (5.6) back into equations (5.4). This results in the following necessary conditions.

$$\begin{aligned} \begin{pmatrix} \boldsymbol{\lambda}_h{}^T \dfrac{\partial \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max}))}{\partial \mathbf{x}} \\ + \boldsymbol{\lambda}_g{}^T \dfrac{\partial \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max}))}{\partial \mathbf{x}} \end{pmatrix} &= \mathbf{0} \\ -\frac{\partial C(\mathbf{s})}{\partial \mathbf{s}} - \tilde{\boldsymbol{\lambda}}_{\hat{h}s} - \boldsymbol{\lambda}_{gs\min} + \boldsymbol{\lambda}_{gs\max} &= \mathbf{0} \\ \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max})) &= \mathbf{0} \\ \boldsymbol{\lambda}_g^T \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max})) &= \mathbf{0} \\ \boldsymbol{\lambda}_g &\geq \mathbf{0} \end{aligned} \quad (5.7)$$

Now compare equations (5.7) to equations (3.3) which are the necessary conditions for optimality for the traditional OPF which minimizes the generation costs. They are the same equations with the additional equation $\mathbf{d} = \mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max})$ enforced "after the fact". Therefore, in order to implement the maximization of social welfare into an existing OPF that minimizes generation costs, one need only add the equation $\mathbf{d} = \mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max})$ to the necessary conditions along with allowing the consumer demand to vary. This is exactly what was done in [4] in order to provide demand control.

6. Consumer Demand Function

The consumer demand in our development is a function of the price paid at the node: $\mathbf{d}_p = \mathbf{D}_p(\mathbf{p}_p)$. This demand function is the inverse of $\frac{\partial B(\bullet)}{\partial \mathbf{d}_p}$. For a more intuitive feel of what this means consider the sample plots of consumer benefit shown Figure 1 and Figure 2.



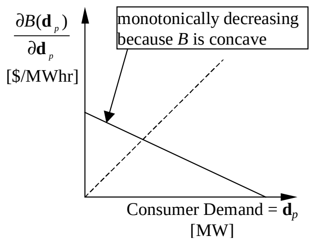


Figure 1 Consumer Benefit **Figure 2 Derivative of *B***

It is important to assume that the consumer benefit function, *B*, be concave and increasing in order to help insure only one social welfare maximum exists (ignoring the constraints). These are good assumptions however. Presumably, the consumer always gains some benefit from more consumption therefore the benefit increases. Even if the consumer does not gain more benefit, he will be able to resell the power on the market. The concavity assumption is valid because an intelligent consumer will always give energy to her most beneficial processes first thereby making the marginal benefit for lower consumption larger.

At the optimal solution from the social welfare perspective, $\frac{\partial B(\bullet)}{\partial \mathbf{d}_p}$ will be the price for each consumer. Thus by taking the inverse of the function (i.e. "flipping" Figure 2 around the dotted line), the consumer demand function will be as shown in Figure 3.

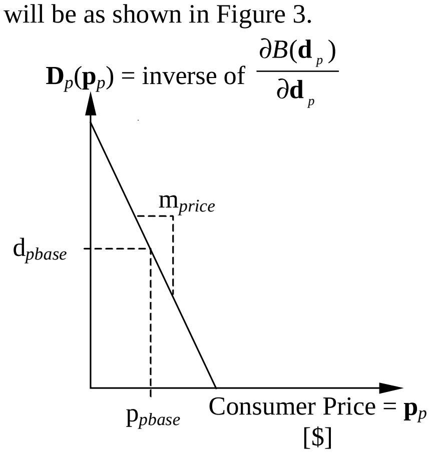


Figure 3 Consumer Demand

This consumer demand function is what must be substituted as the additional "after the fact" equation in the traditional OPF in order to produce the social welfare maximum.

For the examples shown in the remainder of the paper, the consumer demand function for each load will be assumed to be a straight line as shown in Figure 3. The point (p_{base}, d_{base}) and the slope m_{price} will specify the line. Therefore the consumer demand function will be

$$\mathbf{D}_p(\mathbf{p}_p) = (\mathbf{d}_{pbase} + \mathbf{M}_{price}\mathbf{p}_{pbase}) - \mathbf{M}_{price}\mathbf{p}_p \qquad (6.1)$$

where $\mathbf{M}_{price}$ is a diagonal matrix with entries m_{price}. This demand function corresponds to a quadratic consumer benefit function as shown in equation (6.2).

$$B(\mathbf{d}_p) = \mathbf{d}_p^T\left(\mathbf{M}_{price}^{-1}\mathbf{d}_{pbase} + \mathbf{p}_{pbase}\right) - \frac{1}{2}\mathbf{d}_p^T\mathbf{M}_{price}^{-1}\mathbf{d}_p \qquad (6.2)$$

7. Implementation into the OPF

This section will only study how the consumer demand function of equation (6.1) effects the calculations of the Newton's method algorithm. Equation (5.7) will be repeated here to further study how the choice of linear consumer demand functions will effect the solution of these equations.

$$\begin{aligned}
&\left(\begin{array}{l}\boldsymbol{\lambda}_h^T \dfrac{\partial \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max}))}{\partial \mathbf{x}} \\ + \boldsymbol{\lambda}_g^T \dfrac{\partial \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max}))}{\partial \mathbf{x}}\end{array}\right) = \mathbf{0} \\
&-\frac{\partial C(\mathbf{s})}{\partial \mathbf{s}} - \tilde{\boldsymbol{\lambda}}_{\hat{h}s} - \boldsymbol{\lambda}_{gs\min} + \boldsymbol{\lambda}_{gs\max} = \mathbf{0} \\
&\mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max})) = \mathbf{0} \qquad (7.1) \\
&\boldsymbol{\lambda}_g^T \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max})) = \mathbf{0} \\
&\boldsymbol{\lambda}_g \geq \mathbf{0}
\end{aligned}$$

It is first noted that after taking derivatives of **h** and **g** with respect to **x**, no dependence on **s** or **d** is found (as long as **s** and **d** are not functions of **x**, which is normally assumed). Therefore, the choice of the consumer demand function has no effect on the first equation. The only influence comes in the third and fourth equations. The consumer demand function has only changed the demand function from a constant to one dependent on the Lagrange multipliers $\tilde{\boldsymbol{\lambda}}_{\hat{h}d}$, $\boldsymbol{\lambda}_{gd\min}$ and $\boldsymbol{\lambda}_{gd\max}$. This will not impede the OPF algorithm as it will only require a simple function evaluation.

In using Newton's method to solve these nonlinear equations, derivatives of the equations must be determined in order to calculate a Hessian matrix [5]. In order to evaluate how the consumer demand function will effect these equations take the derivatives of the third and fourth equations with respect to $\tilde{\boldsymbol{\lambda}}_{\hat{h}d}$, $\boldsymbol{\lambda}_{gd\min}$ and $\boldsymbol{\lambda}_{gd\max}$.

$$\frac{\partial \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{D})}{\partial \tilde{\boldsymbol{\lambda}}_{\hat{h}d}} = \frac{\partial \begin{bmatrix} \hat{\mathbf{h}}(\mathbf{x}) - \hat{\mathbf{s}} + \hat{\mathbf{d}} \\ \bar{\mathbf{h}}(\mathbf{x}) \end{bmatrix}}{\partial \mathbf{d}} \frac{\partial \mathbf{D}}{\partial \tilde{\boldsymbol{\lambda}}_{\hat{h}d}} = -[\hat{\mathbf{I}}][-\hat{\mathbf{M}}_{price}]$$

$$\frac{\partial \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{D})}{\partial \boldsymbol{\lambda}_{gd\,\min}} = \frac{\partial \begin{bmatrix} \hat{\mathbf{h}}(\mathbf{x}) - \hat{\mathbf{s}} + \hat{\mathbf{d}} \\ \overline{\mathbf{h}}(\mathbf{x}) \end{bmatrix}}{\partial \mathbf{d}} \frac{\partial \mathbf{D}}{\partial \boldsymbol{\lambda}_{gd\,\min}} = +[\hat{\mathbf{I}}][-\tilde{\mathbf{M}}_{price}]$$

$$\frac{\partial \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{D})}{\partial \boldsymbol{\lambda}_{gd\,\max}} = \frac{\partial \begin{bmatrix} \hat{\mathbf{h}}(\mathbf{x}) - \hat{\mathbf{s}} + \hat{\mathbf{d}} \\ \overline{\mathbf{h}}(\mathbf{x}) \end{bmatrix}}{\partial \mathbf{d}} \frac{\partial \mathbf{D}}{\partial \boldsymbol{\lambda}_{gd\,\max}} = -[\hat{\mathbf{I}}][-\overline{\mathbf{M}}_{price}]$$

$$\frac{\partial \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{D})}{\partial \tilde{\boldsymbol{\lambda}}_{\hat{h}d}} = \frac{\partial \begin{bmatrix} \mathbf{s}_{\min} - \mathbf{s} \\ \mathbf{s} - \mathbf{s}_{\max} \\ \mathbf{d}_{\min} - \mathbf{d} \\ \mathbf{d} - \mathbf{d}_{\max} \\ \overline{\mathbf{g}}(\mathbf{x}) \end{bmatrix}}{\partial \mathbf{D}} \frac{\partial \mathbf{D}}{\partial \tilde{\boldsymbol{\lambda}}_{\hat{h}d}} = +[\tilde{\mathbf{I}}][-\hat{\mathbf{M}}_{price}]$$

$$\frac{\partial \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{D})}{\partial \boldsymbol{\lambda}_{gd\,\min}} = +[\tilde{\mathbf{I}}][\tilde{\mathbf{M}}_{price}]$$

$$\frac{\partial \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{D})}{\partial \boldsymbol{\lambda}_{gd\,\max}} = +[\tilde{\mathbf{I}}][-\overline{\mathbf{M}}_{price}]$$

where the variables are defined as

$\hat{\mathbf{I}} =$ matrix with diagonal entries of 1 corresponding to consumer demand variables $\mathbf{d}_p$

$\tilde{\mathbf{I}} =$ matrix with diagonal entries of 1 corresponding to consumer demands $\mathbf{d}_p$ which are at a limit

$\hat{\mathbf{M}}_{price} =$ matrix with diagonal entries of m_{price} corresponding to variables $\mathbf{d}_p$

$\tilde{\mathbf{M}}_{price} =$ matrix with diagonal entries of m_{price} corresponding to variables $\boldsymbol{\lambda}_{gd\,\min}$ related to $\mathbf{d}_p$

$\overline{\mathbf{M}}_{price} =$ matrix with diagonal entries of m_{price} corresponding to variables $\boldsymbol{\lambda}_{gd\,\max}$ related to $\mathbf{d}_p$

The key point to recognize is that the effect of the additional price-dependent equations on the Hessian matrix is limited to diagonal entries. From equation (7) of reference [3], the Hessian matrix for the coupled OPF formulation is shown to have the structure in equation (7.2).

$$\mathbf{W} = \left[\begin{array}{c|c} \mathbf{H} & -\mathbf{J}^{\mathbf{T}} \\ \hline -\mathbf{J} & \mathbf{0} \end{array}\right] \qquad (7.2)$$

The diagonal entries which are added to the Hessian by the price dependent loads will be in the zero matrix in the lower right partition (see Figure 9). Since these diagonals are filled in during the matrix factorization routine, no loss of sparsity occurs from the modification to the OPF algorithm. There also may be possible advantages in ordering the bottom rows as thay are less dense than the top. Furthermore, it is expected that the price dependent load will help with convergence of the OPF. This is because the loads in the system will tend to decrease if the system moves close to a limit due to the price increasing.

It should be recognized however, that if the real power loads are allowed to be functions of both real and reactive power spot prices, then entries will also result on some off-diagonal entries in the lower right zero matrix of equation (7.2). This will be limited to one extra entry per row.

8. Application to a Real Power Spot Market

Showing general ideas

The OPF modifications as discussed in the previous section were implemented into the PowerWorld™ OPF that minimizes fuel costs [7]. As a case for comparison purposes, Figure 4 shows a small six-bus system that has been optimized to minimize fuel costs.

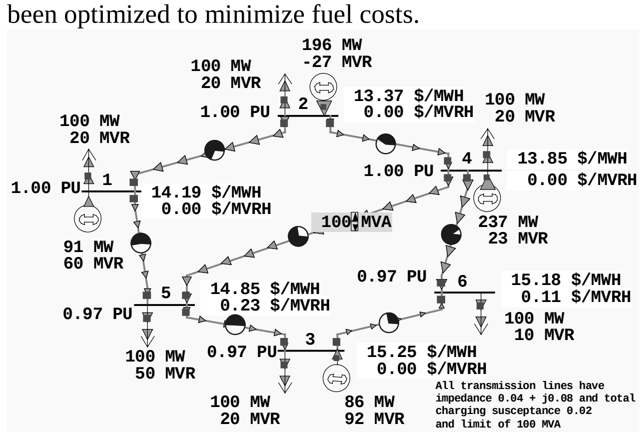


Figure 4 Six-bus system with fuel costs minimized

The costs curves for the generators shown in Figure 4 are

$$C(P_{Gq}) = (105 + 12P_{G1} + 0.012P_{G1}^2) * FuelCost1$$

$$C(P_{G2}) = (96 + 9.6P_{G2} + 0.0096P_{G2}^2) * FuelCost2$$

$$C(P_{G3}) = (105 + 13P_{G1} + 0.013P_{G3}^2) * FuelCost3$$

$$C(P_{Gq}) = (94 + 9.4P_{G1} + 0.0094P_{G4}^2) * FuelCost4$$

with *FuelCostX* = 1.00 \$/BTU for all generators.

The system was then optimized to maximize social welfare by implementing a price dependent demand curve at each bus of

$$d_p(p_p) = d_{pbase}\left(1 + \frac{m_{price}}{d_{pbase}}(p_{pbase} - p_p)\right) = d_{pbase}\left(1 + 10(20 - p_p\right)$$

with d_{pbase} equal to the base demand for the bus. In this example d_{pbase} = 100 MW for each load as shown in Figure 4. This price model means the loads will consume their d_{pbase} if the spot price is p_p = 20 \$/MWH. If the spot price falls below 20 \$/MWH the loads will begin to consume more power, while if the spot price increases the loads will respond by consuming less. This sensitivity to price is encapsulated in the term $\frac{m_{price}}{d_{pbase}}$ that determines the slope of the demand function given in Figure 3. This optimization yields the results shown in Figure 5.

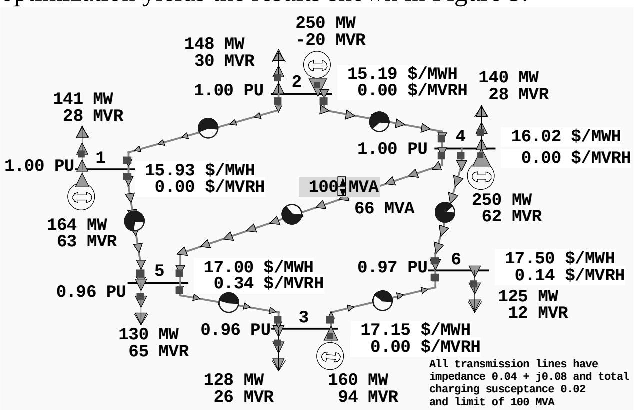


Figure 5 Six-bus system maximizing social welfare

It should be noticed that in Figure 5, the spot prices are all below 20 $/MWH causing the loads to converge more than their d_{pbase} of 100 MW. It should also be noticed that while in Figure 4 the load at every bus was 100 MW, the loads are varied in Figure 5 with the smaller loads at buses with larger marginal costs. These differences are relatively small for this case, but now consider what happens as the system moves toward a transmission line limit. Presently 66 MVA is flowing on the line from bus 4 to bus 5. If this limit were decreased to 40 MVA, then it would be expected that the marginal cost at bus 5 would tend to increase. The consumer would then decrease the demand. This is exactly what happens as is shown in Figure 6.

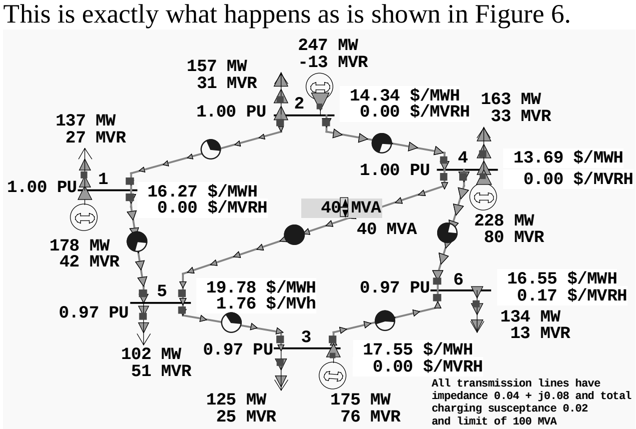


Figure 6 Line Limit has been decreased to 40 MVA

The price increases at bus 5 causing the demand at bus 5 to decrease from 130 MW to 102 MW. It should also be noted that the price decreases at bus 4 causing the demand at bus 4 to increase from 140 MW to 163 MW. The price decreases at bus 4 because the line reaching a limit causes there to be a surplus of cheap power at bus 4.

Now consider another scenario where the cost of fuel increases throughout the entire power system. The line limit is increased back to 100 MVA, but the fuel costs throughout the system are increase by 50%. This should drive the marginal costs of the generators up thus increasing marginal costs throughout the system. Results of this simulation are shown in Figure 7.

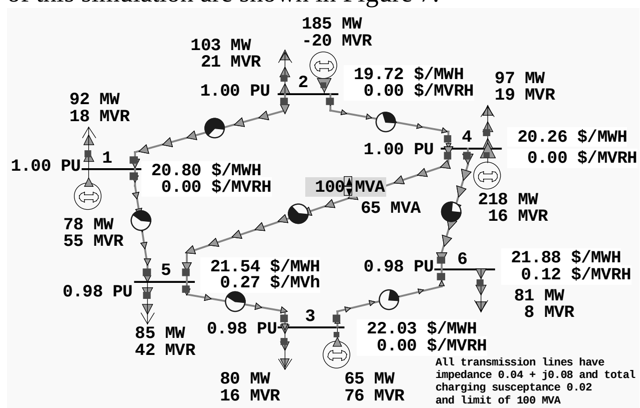


Figure 7 Fuel costs increased by 50%

A larger system – viewing the Hessian

The IEEE 118-bus system was also used to test the algorithms. Cost data was created for this system following the premise that larger units are general cheaper units. The Hessian matrix for the objective of minimizing total generation costs is shown in Figure 8.

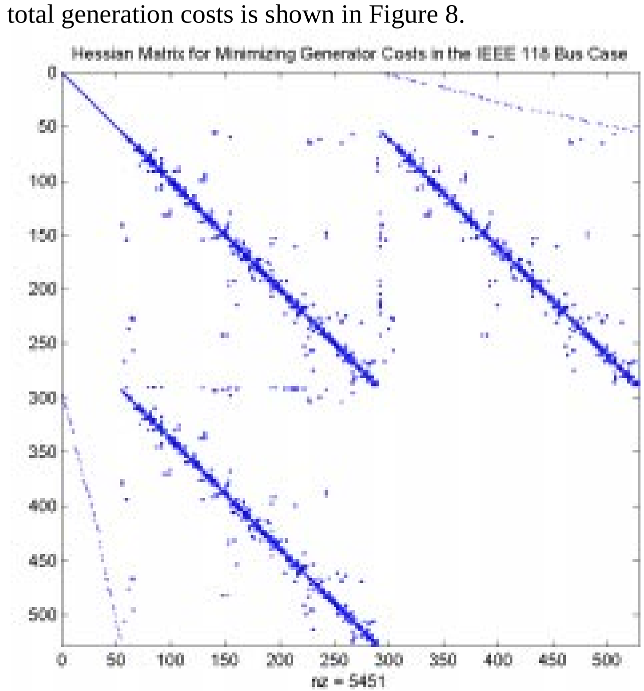


Figure 8 Hessian for minimizing costs

As expected, there is a large zero matrix in the lower right partition. The only differences which will be seen when the price dependent loads are added for determining the social welfare maximization will be on the diagonals of this zero matrix. Figure 9 shows this fact.

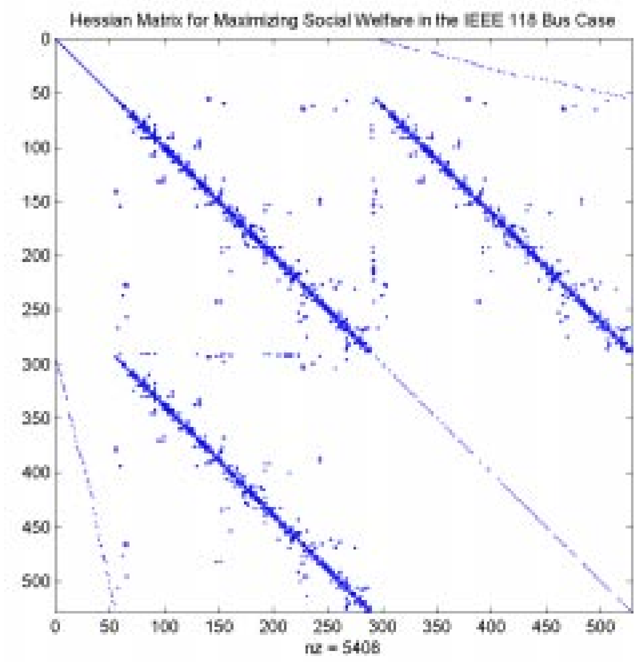


Figure 9 Hessian matrix with price dependent loads

The solution times for solving the OPF for the 118-bus system were very similar for both objective functions. Starting from a flat start for voltages and angles, both systems converged in between 1.3 - 1.5 seconds on several successive runs.

9. A Possible Further Restatement of the Standard OPF

Consider the necessary equations for our alternate approach shown in Equation (5.7). Consider the function $S(\bullet)$ which is the functional inverse of $\frac{\partial C}{\partial \mathbf{s}}(\bullet)$. In other words $\mathbf{p}_s - \frac{\partial C}{\partial \mathbf{s}}(\mathbf{S}(\mathbf{p}_s)) = 0 \quad \forall \mathbf{p}_s$. Then given the equation $-\frac{\partial C(\mathbf{s})}{\partial \mathbf{s}} - \tilde{\boldsymbol{\lambda}}_{\hat{h}s} - \boldsymbol{\lambda}_{gs\min} + \boldsymbol{\lambda}_{gs\max} = \mathbf{0}$ implies that $\mathbf{s} - \mathbf{S}(\tilde{\boldsymbol{\lambda}}_{\hat{h}s} + \boldsymbol{\lambda}_{gs\min} - \boldsymbol{\lambda}_{gs\max}) = \mathbf{0}$.

Therefore, we may rewrite the necessary conditions as

$$\begin{gathered}
\begin{pmatrix} \boldsymbol{\lambda}_h{}^T \dfrac{\partial \mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{D}(\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max}))}{\partial \mathbf{x}} \\ + \boldsymbol{\lambda}_g{}^T \dfrac{\partial \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{D}(\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max}))}{\partial \mathbf{x}} \end{pmatrix} = \mathbf{0} \\
\mathbf{s} - \mathbf{S}(\tilde{\boldsymbol{\lambda}}_{\hat{h}s} + \boldsymbol{\lambda}_{gs\min} - \boldsymbol{\lambda}_{gs\max}) = \mathbf{0} \\
\mathbf{h}(\mathbf{x},\mathbf{s},\mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max})) = \mathbf{0} \\
\boldsymbol{\lambda}_g^T \mathbf{g}(\mathbf{x},\mathbf{s},\mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max})) = \mathbf{0}
\end{gathered} \tag{9.1}$$

which can be simplified to

$$\begin{gathered}
\begin{pmatrix} \boldsymbol{\lambda}_h{}^T \dfrac{\partial \mathbf{h}(\mathbf{x},\mathbf{S}(\tilde{\boldsymbol{\lambda}}_{\hat{h}s} + \boldsymbol{\lambda}_{gs\min} - \boldsymbol{\lambda}_{gs\max}),\mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max}))}{\partial \mathbf{x}} \\ + \boldsymbol{\lambda}_g{}^T \dfrac{\partial \mathbf{g}(\mathbf{x},\mathbf{S}(\tilde{\boldsymbol{\lambda}}_{\hat{h}s} + \boldsymbol{\lambda}_{gs\min} - \boldsymbol{\lambda}_{gs\max}),\mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max}))}{\partial \mathbf{x}} \end{pmatrix} = \mathbf{0} \\
\mathbf{h}(\mathbf{x},\mathbf{S}(\tilde{\boldsymbol{\lambda}}_{\hat{h}s} + \boldsymbol{\lambda}_{gs\min} - \boldsymbol{\lambda}_{gs\max}),\mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max})) = \mathbf{0} \\
\boldsymbol{\lambda}_g^T \mathbf{g}(\mathbf{x},\mathbf{S}(\tilde{\boldsymbol{\lambda}}_{\hat{h}s} + \boldsymbol{\lambda}_{gs\min} - \boldsymbol{\lambda}_{gs\max}),\mathbf{D}(-\tilde{\boldsymbol{\lambda}}_{\hat{h}d} + \boldsymbol{\lambda}_{gd\min} - \boldsymbol{\lambda}_{gd\max})) = \mathbf{0}
\end{gathered} \tag{9.2}$$

In this formulation, consumer demand functions and supplier supply functions can be submitted and the optimum determined.

10. Conclusion

This paper shows that one can make simple modifications to an existing OPF algorithm that minimizes generation costs in order to solve the maximization of social welfare objective of the OPF. This modification is both simple and intuitive and leads to the possibility of simulating a real and reactive power market by asking participants to submit price dependent demand curves. This idea further leads to the possible reformulation of the generation costs into price dependent supply curves. Given a set of price dependent supply curves along with a set of price dependent demand curves, a two sided power market could be simulated.

Appendix

The Power Flow Equations

The first equation is the real power flow equation at bus k, while the second equation is the reactive power flow equation at bus k.

$$V_k \sum_{m=1}^{N} \left[V_m \left[g_{km} \cos(\theta_k - \theta_m) + b_{km} \sin(\theta_k - \theta_m) \right] \right] - S_{Pk} + D_{Pk} = 0$$

$$V_k \sum_{m=1}^{N} \left[V_m \left[g_{km} \sin(\theta_k - \theta_m) - b_{km} \cos(\theta_k - \theta_m) \right] \right] - S_{Qk} + D_{Qk} = 0$$

where g_{km}, b_{km} = Y - Bus (Admittance Matrix) terms
S_{Pk}, S_{Qk} = Real and reactive supply at bus k
D_{Pk}, D_{Qk} = Real and reactive demand at bus k
V_k, θ_k = Voltage magnitude and angle at bus k

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